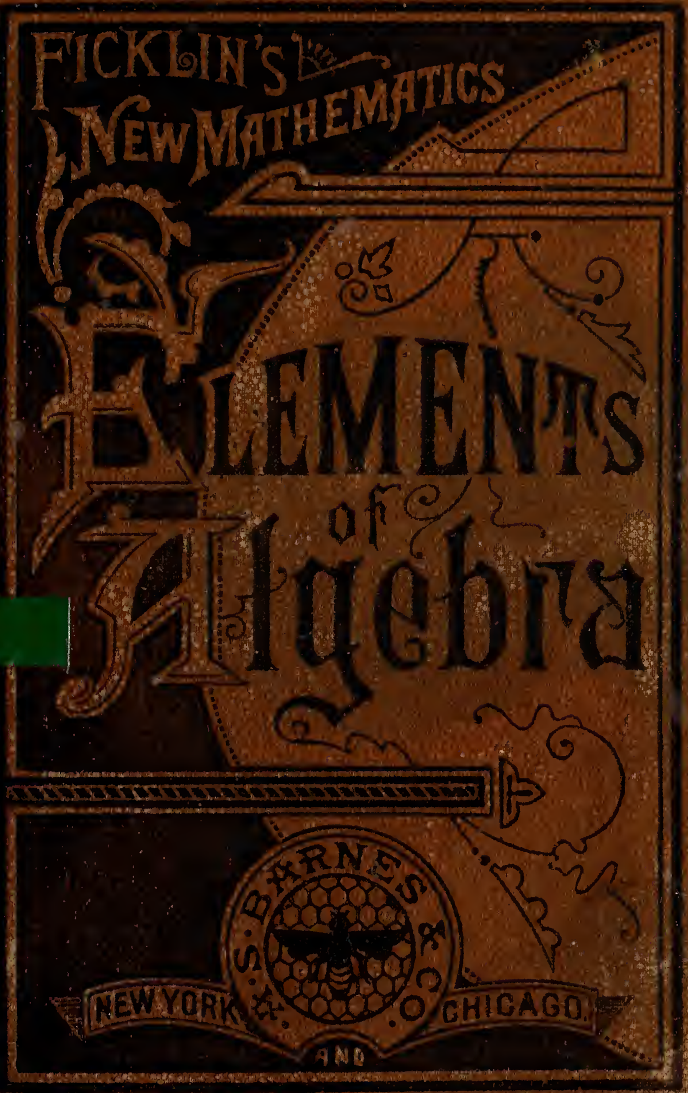

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ELEMENTS of ALGEBRA

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BARNES' NEW SERIES OF MATHEMATICS.

ELEMENTS

OF

ALGEBRA;

DESIGNED FOR

GRAMMAR AND HIGH SCHOOLS, ACADEMIES, ETC.

BY

JOSEPH FICKLIN, PH.D.,
PROFESSOR OF MATHEMATICS AND ASTRONOMY IN THE UNIVERSITY OF THE
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A. S. BARNES & COMPANY,
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PREFACE.

THE present work, as its title indicates, is elementary in its character. The aim has been to prepare a treatise suitable for use, (1) in Public Schools, Academies, and High Schools; (2) in those higher Institutions of learning whose courses of study include elementary work in Algebra; (3) in Schools whose object is to prepare students for admission to the Freshman class of higher Institutions. The author believes that this book will also meet the wants of those students of Physics who require some knowledge of Algebra, but who have not the time to read an exhaustive treatise on the subject.

The student's progress in Algebra depends very largely upon the proper treatment of the four *Fundamental Operations.* Erroneous views at the beginning of any subject are difficult to correct. Hence, just as an architect gives special attention to the foundation of his building, the author has aimed to present this part of the subject with such care that there shall be no occasion to modify or enlarge the foundation here laid. The terms *Addition, Subtraction, Multiplication,* and *Division* in Algebra have a wider meaning than in Arithmetic, and these operations have been so defined as to *include* their arithmetical meaning; so that the beginner is simply called upon to *enlarge* his views of those fundamental operations. Much attention has been given to the explanation of the negative sign, in order to remove, as far as possible, the difficulties which every beginner in Algebra experiences when he comes to use and interpret that sign. Special attention is here called to "A Short Method of Removing Symbols of Aggregation," Art. 76.

On account of their importance, the subjects of *Factoring, Greatest Common Divisor,* and *Least Common Multiple* have been treated at greater length than is usual in elementary works.

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In the treatment of *Fractions,* a method is used which is quite simple, and, at the same time, more general than that usually employed.

Much of the difficulty experienced by the student in connection with *Radical Quantities* entirely disappears, when the roots are expressed by fractional exponents, for the principles and rules applicable to integral exponents may then be used without modification. For this reason fractional exponents are used in this work to a greater extent than usual.

The great object to be attained in the study of Algebra is skill in the use of the *Equation,* the great instrument in mathematical investigations. Nearly every exercise is intended to qualify the student to handle this instrument with ease and rapidity. Hence, the *Equation* is made the chief subject of thought in this work. It is defined near the beginning, and used extensively in every chapter. In addition to this, four chapters are devoted exclusively to the subject of *Equations.* All *Proportions* are equations, and in their treatment they are so considered. By this method all the difficulty connected with the subject of Proportion, as usually treated, disappears.

The Chapter on Logarithms will doubtless be acceptable to many teachers who do not require the student to master Higher Algebra before entering upon the study of Trigonometry.

Such exercises as are not original have been drawn almost exclusively from English works on Algebra. In this respect the "Scholar's Algebra," by Hensley, is the only one deserving special mention.

In conclusion, the author submits this treatise to the examination and judgment of his fellow laborers in the great work of education, feeling assured that they will be quick to discern and appreciate whatever merit it may possess.

COLUMBIA, MO., *August, 1881.*

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ELEMENTS OF ALGEBRA.

CHAPTER I.

INTRODUCTION.

ARTICLE **1. To Measure** a thing is to find the number of times it contains another thing of the same kind taken as a standard. The standard is called *the Unit of Measure*, or *the Measuring Unit*.

2. Quantity is the amount or extent of any thing that can be measured. Thus, 4 hours, 12 bushels, and 8 feet are quantities.

The term quantity is also applied to the characters or expressions by which quantities are represented.

3. Mathematics is the science which treats of the measurement of quantities and their relations to each other.

4. Algebra is that branch of Mathematics in which quantities are represented by letters, or by letters and figures combined, and in which the relations of quantities to each other and the operations to be performed are indicated by *signs*.

The letters, figures, and signs are called *Symbols*.

5. The **Algebraic Language** consists in the use of algebraic symbols.

6. Algebraic Notation consists in representing quantities and indicating operations and relations by means of symbols.

SYMBOLS OF QUANTITY.

7. Known Quantities are those whose values are given. They are represented by numbers or the leading letters of the alphabet.

8. Unknown Quantities are those whose values are not given. They are represented by the final letters of the alphabet.

FUNDAMENTAL OPERATIONS AND THEIR SYMBOLS.

ADDITION.

9. Addition is the process of finding the simplest expression for the aggregate of two or more quantities.

The *Sum* of two or more quantities is the simplest expression for their aggregate, or total value; that is, the *sum* is the result obtained by addition.

To *Add* one quantity to another is to find their sum.

10. The **Sign of Addition**, or the **Plus Sign**, $+$, signifies that the quantity to which it is prefixed is to be *added*. Thus, in $a+b$, which is read "a plus b," the sign $+$ signifies that b is to be added to a. If a represents 8, and b, 5, then $a+b$ is equal to 13.

The sign $+$ is sometimes called the *Positive Sign*.

SUBTRACTION.

11. Subtraction is the process of taking one quantity from another.

The **Subtrahend** is the quantity to be subtracted; the **Minuend** is the quantity from which the subtrahend is to be taken; and the **Difference**, or **Remainder**, is the result obtained by subtraction.

12. The **Sign of Subtraction**, or the **Minus Sign**, —, signifies that the quantity to which it is prefixed is to be subtracted. Thus, in $a - b$, which is read "a minus b," the sign — signifies that b is to be subtracted from a. If a represents 8, and b, 5, then $a - b$ is equal to 3.

The sign — is sometimes called the *Negative Sign*.

THE DOUBLE SIGNS.

13. To indicate that a quantity is to be added or subtracted, the sign ± or the sign ∓ is prefixed to it. The sign ± is read "plus or minus," and ∓ is read "minus or plus."

MULTIPLICATION.

14. To **Multiply** one quantity by another is to find a third quantity which has the same relation to the first that the second has to a unit $(+1)$.

The **Multiplier** is the quantity by which some other quantity is to be multiplied; the **Multiplicand** is that which is to be multiplied; and the **Product** is the result obtained by multiplication. Thus, if it be required to multiply 5 by 4, 5 is the multiplicand, 4 is the multiplier, and 20 is the product.

The **Product of Three Quantities** is the result obtained by multiplying the product of two of them by the third; the **Product of Four Quantities** is the result obtained by multiplying the product of three of them by the fourth; and so on.

A **Continued Product** is the product of three or more quantities.

15. The **Signs of Multiplication** are × and $\cdot$, each of which signifies that the quantity which precedes it is to be *multiplied* by the one which follows it. Thus, in each of the expressions $a \times b$ and $a \cdot b$, the sign of multiplication signifies that a is to be multiplied by b, and each is read, "a into b," or "a multiplied by b."

The sign of multiplication is often omitted. Thus, $3ab$ is equivalent to $3 \times a \times b$.

DIVISION.

16. To **Divide** one quantity by another is to find a third quantity, such that if it be multiplied by the second, the product will be equal to the first.

The **Divisor** is the quantity by which some other quantity is to be divided; the **Dividend** is that which is to be divided; and the **Quotient** is the result obtained by division. Thus, if it be required to divide 20 by 4, 20 is the dividend, 4 is the divisor, and 5 is the quotient.

17. The **Sign of Division,** $\div$, signifies that the quantity which precedes it is to be *divided* by the one which follows it. Thus, in $a \div b$, which is read, "a by b," or "a divided by b," the sign $\div$ signifies that a is to be divided by b.

The expression $\frac{a}{b}$ is equivalent to $a \div b$.

SYMBOLS OF AGGREGATION.

18. The **Symbols of Aggregation** are the *Parenthesis*, ( ); the *Vinculum*, ——; the *Brackets*, [ ]; and the *Braces*, { }.

When an expression is embraced by a symbol of aggregation, each operation indicated in that expression is to be regarded as performed, and the symbol is to be regarded as expressing the result.

Thus, $(a - b)\,c$, which is read "parenthesis $a - b$ into

c," signifies that the result obtained by subtracting b from a is to be multiplied by c.

The expressions $(a-b)\,c$, $\overline{a-b}\times c$, $[a-b]\,c$, and $\{a-b\}c$ are equivalent.

The vinculum is sometimes placed in a vertical position. Thus, $\left.\begin{array}{r} a \\ -b \end{array}\right| c$ is equivalent to $\overline{a-b}\times c$.

SYMBOLS OF RELATION.

19. The **Sign of Equality,** $=$, signifies that the quantities between which it is placed are equal. Thus, in $xy=a+b$, which is read "x into y equals a plus b," or "x into y is equal to a plus b," the sign $=$ signifies that the product of x and y is equal to the sum of a and b.

20. The **Signs of Inequality** are $>$ and $<$, each of which signifies that the quantities between which it is placed are *unequal,* the opening being turned toward the greater. Thus, in $x+y>x-y$, which is read "x plus y is greater than x minus y," the sign $>$ signifies that the sum of x and y is greater than their difference ; and in $x-y<x+y$, which is read "x minus y is less than x plus y," the sign $<$ signifies that the difference between x and y is less than their sum.

EQUATION AND INEQUALITY.

21. An **Equation** consists of two expressions connected by the sign of equality. Thus, $x-y=a+b$ is an equation.

The **First Member** of an equation is the expression on the left of the sign of equality, and

The **Second Member** is the expression on the right of the sign.

Thus, in the equation $x - y = a + b$, $x - y$ is the first member, and $a + b$, the second member.

22. An **Inequality** consists of two expressions connected by a sign of inequality, and its members are named as those of an equation.

FACTORS.

23. When one quantity is the product of two or more other quantities, each of the latter is called a **Factor** of that product. Thus, 5, a, b, and c are factors of $5abc$.

24. A **Numerical Factor** is a factor which is expressed by a figure or figures. Thus, in $12abc$, 12 is a numerical factor.

25. A **Literal Factor** is a factor which is expressed by a letter or letters. Thus, in $12abc$, a, b, and c are literal factors.

26. When a product contains one factor which is numerical, and another which is literal, the former factor is called the **Coefficient** of the latter. Thus, in $12abc$, 12 is the coefficient of abc; and in $-7a$, -7 is the coefficient of a.

The term *coefficient* is sometimes applied to any factor. Thus, in $12abc$, $12a$ is the coefficient of bc; and in abc, a is the coefficient of bc, and ab is the coefficient of c.

If no numerical coefficient is expressed, 1 is understood. Thus, abc is equivalent to $1abc$.

POWERS AND ROOTS.

27. A **Power** of a quantity is either the quantity itself or the product of factors, each of which is identical with it. Thus, 5, 5×5 or 25, and $5 \times 5 \times 5$ or 125, are powers of 5.

A quantity is said to be *involved*, or *raised* to a power, when any power of it is found.

28. The **Root,** or **Base,** of a quantity, regarded as a power, is either the quantity itself, or a factor which being involved will produce that quantity. Thus, 5 is a root of 5, 25, and 125, for $5 = 5$, $5 \times 5 = 25$, and $5 \times 5 \times 5 = 125$.

29. The **Degree** of a power is determined by the number of times the quantity regarded as its base must be used to produce the power. Thus, 5 is the *first* power of 5, 25 is its *second* power, and 125 is its *third* power.

The *second* power of a quantity is sometimes called its *square,* because the area of a square is equal to the second power of one of its *sides;* and the *third* power of a quantity is sometimes called its *cube,* because the volume of a cube is equal to the third power of one of its *edges.*

30. An **Exponent** is a number placed at the right of, and a little above, the base of a power; and when it is an integer, it denotes the degree of the power. Thus, a^2, b^3, c^4 indicate respectively the *square* of a, the *cube* of b, and the *fourth* power of c.

If no exponent is expressed, 1 is understood. Thus, a is equivalent to a^1.

The expression a^2 may be read in any of the following ways: "The second power of a," "a second power," "a second," "the square of a," or "a squared"; a^3 may be read in a similar way; a^4 may be read, "the fourth power of a," "a fourth power," or "a fourth"; and so on.

31. The **Degree** of a root is the same as that of the power of which it is considered as the base. Thus, 5 is the *first root* of 5; 3 is the *second,* or *square* root of 9, for $3^2 = 9$; and 5 is the *third,* or *cube* root of 125, for $5^3 = 125$.

32. The **Radical Sign,** $\sqrt{\ }$, is the character used to indicate that a root is to be extracted.

33. The **Index** of a root is the number placed over the radical sign to denote the degree of the root. Thus, $\sqrt[3]{27}$,

$\sqrt[4]{64}$, $\sqrt[5]{32}$ indicate respectively the cube root of 27, the fourth root of 64, and the fifth root of 32.

The *square root* is usually indicated by the radical sign without the index.

ALGEBRAIC EXPRESSIONS.

34. A **Monomial** is an expression which does not contain parts connected by the sign + or the sign —. Thus, $5ab$ is a monomial.

A monomial is sometimes called a *Simple Expression*, or a *Term*.

35. A **Polynomial** is an expression which contains parts connected by the sign + or the sign —, and each of these parts with its sign is called a *Term*. Thus, $a^2 - 2ab + b^2$ is a polynomial, and a^2, $-2ab$, and $+b^2$ are its terms.

When a term has no sign expressed, the sign + is understood.

A polynomial is sometimes called a *Compound Expression*.

36. A **Binomial** is a polynomial containing only *two* terms. Thus, $a^2 - b^2$ is a binomial.

37. A **Trinomial** is a polynomial containing only *three* terms. Thus, $a^2 + 2ab + b^2$ is a trinomial.

38. A polynomial may be formed, when its terms are given, by simply writing them one after the other, and giving to each its own sign. Thus, if a, $-b$, c, and $-d$ are the terms, the polynomial is $a - b + c - d$.

39. A term is simple or compound. Thus, in the expression $a^2 - 2ab + b^2 - (c - d)$, a^2, $-2ab$, and $+b^2$ are simple terms, and $-(c - d)$ is a compound term.

40. A **Positive Term** is a term whose sign is +.

41. A **Negative Term** is a term whose sign is —.

42. A **Dimension** of a term is one factor of it expressed by a single letter.

43. The **Degree** of a term is equal to the number of its dimensions. Hence, the degree of a term is equal to the sum of the exponents of its letters. Thus, $5a^3b^4c$ is of the eighth degree.

44. Similar Terms are those whose literal parts are *identical.* Thus, $5a^2b$ and $10a^2b$ are similar.

45. Dissimilar Terms are those whose literal parts are *different.* Thus, $3a^2b$ and $5ab^2$ are dissimilar.

An exception must be made in cases where literal factors are considered as coefficients. Thus, ax^2 and bx^2 are similar, if a and b are considered as coefficients.

46. A **Homogeneous Polynomial** is a polynomial whose terms are all of the same degree. Thus, $a^3 + 3a^2b + 5abc$ is homogeneous.

47. The **Numerical Value** of an algebraic expression is the number obtained by substituting for each letter the number which it represents, and then performing the indicated operations. Thus, if $a = 3$ and $b = 4$, the numerical value of $a^2 + 2ab + b^2$ is 49.

EXERCISES IN READING THE ALGEBRAIC LANGUAGE.

48. Read:

1. $5a + 9c - 3d$.

Ans. Five a plus nine c minus three d.

2. $3a^2 + 4ab - 6b^4$.

Ans. Three a square plus four ab minus six b fourth.

3. $a(c + d - e)$.

Ans. a into the parenthesis c plus d minus e.

4. $a^3(c^4 - d^5 + e^6)^8$.

Ans. a cube into the eighth power of c fourth minus d fifth plus e sixth.

5. $17x - 3 = 45 + 11x.$

6. $3a + x - 5b + 2 = 7b - a + c + b.$

7. $2(x - 3)^2 = 3(x - 4).$

8. $\dfrac{3a + x}{x} - 5 = \dfrac{6}{x}.$

9. $(3 + x)(5 + x) - 48 = \dfrac{99}{5} + x^2.$

10. $\dfrac{a + b}{cd} > p(m + n - x - y)^2.$

11. $\sqrt{x} - \sqrt{y} < n\sqrt{a} + \sqrt{b}.$

12. $\sqrt{x - y} < n\sqrt{a + b}.$

EXERCISES IN WRITING THE ALGEBRAIC LANGUAGE.

49. Write in the algebraic language:

1. Fifteen c minus seventeen d plus eighteen e.
Ans. $15c - 17d + 18e.$

2. Eighty-one b minus thirteen c plus fourteen e minus nineteen f. *Ans.* $81b - 13c + 14e - 19f.$

3. Five a square plus six ab minus seven b cube.
Ans. $5a^2 + 6ab - 7b^3.$

4. Five a square b minus eight ab square plus six b fifth.
Ans. $5a^2b - 8ab^2 + 6b^5.$

5. Nine a fourth b cube plus four a cube b fourth minus three a square b plus seven ab cube minus five b fourth.
Ans. $9a^4b^3 + 4a^3b^4 - 3a^2b + 7ab^3 - 5b^4.$

6. a square plus twice bc minus c square plus d is equal to mx. *Ans.* $a^2 + 2bc - c^2 + d = mx.$

7. a plus b divided by cd is less than p into the parenthesis m plus n plus x minus y.

Ans. $\dfrac{a + b}{cd} < p(m + n + x - y).$

8. The square root of a plus b is greater than the cube root of x minus y. *Ans.* $\sqrt{a + b} > \sqrt[3]{x - y}.$

9. The square root of a, plus b is greater than the cube root of x, minus y. *Ans.* $\sqrt{a}+b>\sqrt[3]{x}-y$.

10. The square root of a square minus b cube equals x minus the cube root of x plus a.

Ans. $\sqrt{a^2-b^3}=x-\sqrt[3]{x+a}$.

EXERCISES IN FINDING NUMERICAL VALUES.

50. If $a=8$, $b=6$, $c=4$, $d=2$, $m=3$, $n=1$, find the numerical value of each of the following expressions:

1. $a^2+2ab+b^2$. *Ans.* 196.
2. $a^2+b^2+c^2+d^2$. *Ans.* 120.
3. $a+b\times c-d$. *Ans.* 30.
4. $(a+b)(c-d)$. *Ans.* 28.
5. $\dfrac{a^2+b^2+c^2+d^2}{3(a+b+c+d)}$. *Ans.* 2.
6. $\dfrac{(6a^2n-4m^3d)(m^2-7n)}{mn}$. *Ans.* 112.
7. $\dfrac{a(5an+4b+d)}{a+b+c+d+m+n}$. *Ans.* 22.
8. $a\left(\dfrac{5ad^2-3c^2}{cd^2}\right)$. *Ans.* 56.
9. $\dfrac{am}{c}+\dfrac{d^4-n}{3(m^2-b)}$. *Ans.* $7\frac{2}{3}$.
10. $\left[\dfrac{(a+b-2d)(a-b+2d)}{3m-2d(1+n)+n}\right](m-d)$. *Ans.* 30.
11. $\dfrac{\sqrt{c}}{\sqrt[3]{b+d}}+\sqrt{\dfrac{m+n}{c}}$. *Ans.* 2.
12. $\dfrac{\sqrt[3]{a}+\sqrt{ad}}{\sqrt{m-d}}$. *Ans.* 6.

LOGICAL TERMS.

51. A **Problem** is the statement of something to be done.

The *Solution* of a problem consists in doing what the problem requires.

52. An **Hypothesis** is a supposition.

53. A **Proposition** is the statement of a real or supposed truth.

54. Propositions are distinguished by the following names: Definition, Axiom, Theorem, Corollary, Scholium, Postulate.

55. A **Definition** is such a description of an object as distinguishes it from all other objects.

A definition should be both *inclusive* and *exclusive;* that is, it should *include* the thing defined, and *exclude* every thing else.

56. An **Axiom** is a self-evident proposition.

57. A **Theorem** is a proposition, the truth of which is made to appear by a course of reasoning called a *Demonstration,* or *Proof.*

58. A **Formula** is a theorem expressed in algebraic language.

59. A **Lemma** is an auxiliary theorem used in the demonstration of another theorem, or in the solution of a problem.

60. A **Corollary** is an obvious consequence of one or more of the preceding principles.

61. A **Scholium** is a remark.

62. A **Postulate** assumes the possibility of solving some problem.

63. Abbreviations.

Prop. is an abbreviation for *Proposition.*
Th. “ “ “ *Theorem.*
Cor. “ “ “ *Corollary.*
Sch. “ “ “ *Scholium.*
G. C. D. “ “ “ *Greatest Common Divisor.*
L. C. M. “ “ “ *Least Common Multiple.*
A. P. “ “ “ *Arithmetical Progression.*
G. P. “ “ “ *Geometrical Progression.*
Log. “ “ “ *Logarithm.*
$\therefore$ “ “ “ *Hence, Therefore,* or *Consequently.*

AXIOMS.

64. 1. *The whole is equal to the sum of all its parts.*

2. *If each of two quantities is equal to a third quantity, they are equal to each other.*

3. *If two equal quantities be increased, diminished, multiplied, or divided by the same quantity, or by equal quantities, the results will be equal.*

4. *Like powers of two equal quantities are equal.*

5. *Like roots of two equal quantities are equal.*

6. *If the members of an equation be made to exchange places, the equality will not be destroyed.* Thus, if $A = B$, then $B = A$.

CHAPTER II.

FUNDAMENTAL OPERATIONS.

ADDITION.

65. The sum of two or more quantities has been defined to be *the simplest expression for their aggregate*, or *total value.* This definition is complete; but it may not convey to the mind of the beginner the full meaning of the word *sum.* Its full meaning may be learned from the following definition:

The **Sum** of two or more quantities is the expression obtained by connecting them in the same manner as terms are connected to form a polynomial (**38**), and reducing the result to its simplest form. Thus, the sum of $5a$ and $2a$ is $7a$, for $5a + 2a = 7a$; and the sum of $5a$ and $-2a$ is $3a$, for $5a - 2a = 3a$.

66. The **Absolute Value** of a quantity is the number represented by that quantity taken independently of its sign. Thus, the *absolute* value of -8 is 8. Two quantities are *equal* when they have the same absolute value and like signs. Two quantities may have the same absolute value and be unequal. Thus, $+5$ and -5 have the same absolute value, but they are not equal. Such quantities as $+5$ and -5 are sometimes said to be *numerically equal.*

67. In Algebra, addition does not necessarily imply augmentation in an arithmetical sense; nevertheless, for the sake of uniformity, the word *sum* is used to denote the result. Sometimes, when there might be an uncertainty on the point, the phrase *algebraic sum* is used to distinguish such a result from the *arithmetical sum*, which would be obtained by the addition of the absolute values of the terms considered.

68. To find the sum of two similar monomials.

1. Let it be required to find the sum of $2a$ and $3a$.

It is evident that $2a + 3a = 5a$.

2. Let it be required to find the sum of $-2a$ and $-3a$.

Connecting the terms, the result is

$$-2a - 3a.$$

When negative terms occur in an expression, their absolute values are to be taken, in succession, from some other quantity, either in the expression before us, or when the occasion arises. But to take away $2a$ and $3a$, in succession, is the same in effect as to take away, by one subtraction, their aggregate $5a$;

$$\therefore \quad -2a - 3a = -5a.$$

3. Let it be required to find the sum of $5a$ and $-2a$.

$$5a - 2a = 3a.$$

4. Let it be required to find the sum of $2a$ and $-5a$.

$$2a - 5a = 2a - 2a - 3a = 0 - 3a = -3a.$$

Here it is required to subtract $5a$ from $2a$. Taking $2a$ from $2a$, the remainder is 0, and there still remains $3a$ to be subtracted, as the sign $-$ in the result indicates.

RULES.

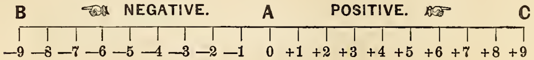
I. *To find the sum of two similar terms with like signs, add their absolute values and prefix the common sign to the result.*

II. *To find the sum of two similar terms with unlike signs, subtract the less absolute value from the greater, and prefix to the result the sign of that term which has the greater absolute value.*

Cor. To *add* a negative quantity is the same in effect as to subtract a positive quantity of the same absolute value. Thus, the sum of 5 and -2 is 3, and if $+2$ be subtracted from 5 the result will be 3.

Sch. 1. The signs + and — have been defined as indicating addition and subtraction respectively; but they have a wider meaning. Sometimes they are used to indicate the *character,* or *quality,* of the things under consideration. In business transactions, a sum of money may be *gained* or it may be *lost;* in a question of chronology, a date *before* a given epoch may have to be distinguished from a date *after* that epoch; in a question of position, a distance measured to the *north* of a certain point may have to be distinguished from a distance measured to the *south* of it; and so on. These pairs of related magnitudes the algebraist distinguishes by means of the signs + and —. Thus, he may denote by $+a$ a gain of a dollars, and then he will denote by $-a$ a loss of the same extent.

Sch. 2. The relation between positive and negative quantities is exhibited to the eye in the following diagram, where the distance from the zero point A to any point in the indefinite line BC is considered positive or negative, according as that point is on the right or on the left of A:



Sch. 3. The following illustrations show how the results obtained by the two preceding rules may be interpreted:

1. Suppose a man starts from A in the line BC, and travels 3 miles toward the right, and then 5 miles further in the same direction; his final distance from A will be 8 miles in the positive direction. This may be considered as an interpretation of the $+8$ obtained by adding $+3$ to $+5$.

2. Suppose a man starts from A, and travels first 3 miles toward the left, and then 5 miles further in the same direction; his final distance from A will be 8 miles in the negative direction. This may be considered as an interpretation of the -8 obtained by adding -3 to -5.

3. Suppose a man starts from A, and travels first 3 miles toward the left, and then turns and travels 5 miles toward the right; his final distance from A will be 2 miles in the positive direction. This may be considered as an interpretation of the $+2$ obtained by adding -3 to $+5$.

4. Suppose a man starts from A, and travels first 3 miles toward the right, and then turns and travels 5 miles toward the left; his final distance from A will be 2 miles in the negative direction. This may be considered as an interpretation of the -2 obtained by adding $+3$ to -5.

69. To find the sum of three or more similar monomials.

1. Let it be required to find the sum of $2a$, $3a$, and $5a$.

It is evident that $2a+3a+5a=10a$.

The terms may be placed in a vertical column and added, as in Arithmetic.

$$\begin{array}{r} 2a \\ 3a \\ \underline{5a} \\ 10a. \end{array}$$

2. Let it be required to find the sum of $6a$, $10a$, $-5a$, and $-2a$.

Connecting the terms, the result is

$$6a+10a-5a-2a.$$

Now to take $5a$ and $2a$, in succession, from a quantity is the same in effect as to take away, by one subtraction, their aggregate $7a$;

$$\begin{array}{l} \;\;6a-5a \\ \underline{10a-2a} \\ 16a-7a=9a. \end{array}$$

$$\therefore\;\; 6a+10a-5a-2a=16a-7a=9a.$$

3. Let it be required to find the sum of $15a$, $3a$, $6a$, $-12a$, $-8a$, and $-9a$.

$$\begin{array}{r}
15a - 12a \\
3a - 8a \\
6a - 9a \\
\hline
24a - 29a = -5a.
\end{array}$$

All the terms may be placed in one column, as in the margin, and it is evident that the result will be the same in whatever order they may be combined. In this case, if we begin at the top of the column and proceed downward, the steps in the addition are as follows: $15a$, $18a$, $24a$, $12a$, $4a$, $-5a$. The literal part may be omitted until the final result is obtained. Thus, 15, 18, 24, 12, 4, $-5a$.

$$\begin{array}{r}
+15a \\
+3a \\
+6a \\
-12a \\
-8a \\
-9a \\
\hline
-5a.
\end{array}$$

4. Find the sum of $-2a$, $-3a$, $-5a$, $-6a$, and $-7a$.

$$\begin{aligned}
-2a-3a-5a-6a-7a &= -5a-11a-7a \\
&= -16a-7a \\
&= -23a \text{ } (\mathbf{68}).
\end{aligned}$$

$$\begin{array}{r}
-2a \\
-3a \\
-5a \\
-6a \\
-7a \\
\hline
-23a.
\end{array}$$

RULES.

I. 1. *Reduce the positive similar terms to one term by addition.*

2. *In like manner reduce the negative similar terms to one term.*

3. *Then reduce the result to one term by the second rule of Art.* **68.**

II. *Reduce any two terms to one* (**68**)*; then reduce the resulting term and a third term to one, and so on until all the terms have been used. The final result will be the required sum.*

EXERCISES.

Find the sum of

1. $13a,\ 5a,\ -4a,\ -17a,\ 21a.$ *Ans.* $18a.$
2. $6b,\ -18b,\ -5b.$ *Ans.* $-17b.$
3. $-3x,\ -2x,\ -8x,\ 6x,\ -3x,\ 15x.$ *Ans.* $5x.$
4. $7ax,\ 3ax,\ -5ax,\ -8ax,\ 5ax,\ -13ax.$ *Ans.* $-11ax.$
5. $6ac^2,\ -5ac^2,\ 7ac^2,\ -3ac^2,\ 12ac^2.$ *Ans.* $17ac^2.$
6. $9a^3xy,\ -5a^3xy,\ 13a^3xy,\ -17a^3xy.$ *Ans.* $0.$
7. $5a,\ -7a,\ a,\ -3a.$ *Ans.* $-4a.$
8. $5x^2,\ -7x^2,\ x^2,\ 2x^2,\ 10x^2,\ -9x^2.$ *Ans.* $2x^2.$

70. To find the sum of expressions, all of whose terms are dissimilar.

Let it be required to find the sum of $a+b$ and $c+d-e$.

Since these expressions have no similar terms, their sum is

$$a+b+c+d-e \ (\mathbf{65}).$$

The same result may be obtained thus:

Adding $c+d$ to $a+b$, the result is $a+b+c+d$. But too much by e has been added to $a+b$; hence, in order to obtain the correct sum, e must be subtracted from $a+b+c+d$; therefore,

$$a+b+c+d-e$$

is the required sum.

RULE.

Write the given expressions one after another, giving to each term its own sign.

Cor. The value of an expression is the same, in whatever order its terms may be written. Thus,

$$a+b-c = a-c+b = b+a-c = b-c+a = -c+b+a = -c+a+b.$$

Sch.—The terms of an expression are usually written in alphabetical order, unless there is a special reason for writing them in a different order.

EXERCISES.

Find the sum of

1. a and $-b$. *Ans.* $a-b$.
2. $-a$ and b. *Ans.* $-a+b$.
3. $-a$ and $-b$. *Ans.* $-a-b$.
4. $a-b$, $c+d$, and $e-f$. *Ans.* $a-b+c+d+e-f$.
5. $a-b-c$, $-d+e-f$, and $-g-h$. *Ans.* $a-b-c-d+e-f-g-h$.
6. $-x-a-b$, $-y-c-d$, $e+f-g$. *Ans.* $-a-b-c-d+e+f-g-x-y$.

71. To find the sum of expressions, some of whose terms are similar.

Let it be required to find the sum of $2a+3b-c$, $3a-2b+2c-e$, and $a+b+c+2e$.

Connecting these expressions thus,

$$2a+3b-c+3a-2b+2c-e+a+b+c+2e,$$

and reducing the result to its simplest form, the sum is found to be

$$6a+2b+2c+e.$$

The same result may be obtained by placing the similar terms in the same column, and reducing each column separately, thus:

$$\begin{array}{rrrrr}
2a & +3b & -c & & \\
3a & -2b & +2c & -e & \\
a & +b & +c & +2e & -f \\
\hline
6a & +2b & +2c & +e & -f.
\end{array}$$

RULE.

1. *Write the given expressions so that similar terms shall fall in the same column.*

2. *Reduce each column of similar terms* (**69**), *and to the result connect the other terms by their signs.*

EXERCISES.

Find the sum of

1. $2a + 3b - c + d$ and $3a - 2b + 2c - e$.
Ans. $5a + b + c + d - e$.

2. $a - 3b + c$, $2a - 3b - 5c$, $-3a + 5b - 6c$, $7a - 6b - 9c$.
Ans. $7a - 7b - 19c$.

3. $2x - 5y + 6$, $2x - 3y - 9$, $-3x + 8y - 7$, $5x - 3y + 2$.
Ans. $6x - 3y - 8$.

4. $7ab + 8bc - 5$, $2ab - 3bc + 9$, $-7ab + 4$, $8bc - 17$.
Ans. $2ab + 13bc - 9$.

5. $3 - a + b$, $2a - 5 + 9b$, $3b - 7 + 5a$, $2 - 9b$, $6a + 5$.
Ans. $12a + 4b - 2$.

6. $7ab^2 - cd$, $8cd - 5ab^2$, $7ab^2 + 16cd$, $-8ab^2 + 15cd$.
Ans. $ab^2 + 38cd$.

7. $30ab + 15bc$, $3bc + 2ac$, $16ac + 5ab$, $7ab - 21bc$, $19bc - 3ac$, $7ac - 15bc$, $16ac - 15ab$.
Ans. $27ab + bc + 38ac$.

8. $2a^2 + 3b^2 - 5cd$, $-4a^2 - 2b^2 + 6ce$, $6a^2 + 9b^2 - 5de$, $7a^2 - 3b^2 + 5de$, $8a^2 + b^2 + ce$.
Ans. $19a^2 + 8b^2 - 5cd + 7ce$.

9. $x^2 - 2y^2$, $3x^2 + 5y^2$, $4x^2 - 6y^2 + z^2$, $8y^2 - 2z^2$, $-7x^2 + 5y^2 - 6z^2$.
Ans. $x^2 + 10y^2 - 7z^2$.

10. $4abc - 2yz^2 + 3y^3$, $5abc + 6yz^2 - 2y^3$, $-4abc + yz^2 - 9y^3$, $-7abc - 3yz^2 + y^3$, $6abc + yz^2 - 9y^3$.
Ans. $4abc + 3yz^2 - 16y^3$.

11. $3a - 2b$, $4a + 7b$, $2a + 3b$, $a - 5b$.
Ans. $10a + 3b$.

12. $9a^2 + 7b^2$, $-3a^2 + 4b^2$, $a^2 + b^2$, $4a^2 - 12b^2$.
Ans. $11a^2$.

13. $a + b + c$, $3a + 2b + 3c$, $-4a + 7b - c$, $2b + 5c$.
Ans. $12b + 8c$.

14. $x - y - z,\ y - x - z,\ z - x - y,\ x + y + z.$ *Ans.* $0.$

15. $3a^2 - 4ab + 6b^2,\ 7ab - a^2 - b^2,\ 2a^2 - 3ab - 4b^2,\ 4a^2 + ab - b^2.$ *Ans.* $8a^2 + ab.$

16. $2x^4 - 7x^2 + 3,\ -4x^3 + 6x^2 - 2x + 7,\ x^4 - 2x^3 - 4x,\ 6x^3 - 9x - 12.$ *Ans.* $3x^4 - x^2 - 15x - 2.$

SYMBOLS OF AGGREGATION PRECEDED BY THE SIGN OF ADDITION.

72. TH. *If a symbol of aggregation preceded by the sign + occurs in an expression, the value of the expression is not changed by omitting that symbol.*

Thus, $a + b + (c - d) = a + b + c - d$ (**70**).

COR. *The value of an expression is not changed by introducing a symbol of aggregation preceded by the sign +.*

Thus, $a+b+c-d = a + b + (c - d) = a + (b+c-d) = a - d + (b + c).$

SUBTRACTION.

73. The **Negative** of a quantity is the result obtained by changing the sign of each of its terms. Thus, the negative of $+a$ is $-a$, the negative of $-a$ is $+a$, and the negative of $-a + b - c$ is $+a - b + c$.

74. To find the difference between two quantities.

1. Let it be required to subtract b from $a + b$.

Taking $+b$ from $a + b$, that is, removing $+b$ from the expression $a + b$, the result is a.

$$\begin{array}{r} a + b \\ - b \\ \hline a. \end{array}$$

The same result may be obtained by *adding* $-b$ to $a + b$ (**71**).

2. Let it be required to subtract $-b$ from $a-b$.

Taking $-b$ from $a-b$, that is, removing $-b$ from the expression $a-b$, the result is a.

The same result may be obtained by *adding* $+b$ to $a-b$.

$$\begin{array}{r} a-b \\ \underline{\quad +b} \\ a. \quad \end{array}$$

3. Let it be required to subtract $c-d$ from $a+b$.

Taking c from $a+b$, the result is $a+b-c$. This result is too small by d, for c is greater by d than the quantity to be subtracted. Therefore, to obtain the correct remainder, d must be added to $a+b-c$.

$$\therefore \quad a+b-(c-d) = a+b-c+d.$$

$$\begin{array}{c} a+b \\ \underline{c-d} \\ a+b-c+d. \end{array} \qquad \begin{array}{c} a+b \\ \underline{-c+d} \\ a+b-c+d. \end{array}$$

The same result may be obtained by adding $-c+d$ to $a+b$.

RULE.

Add the negative of the subtrahend to the minuend.

Cor. 1. To subtract a positive quantity is the same in effect as to add a negative quantity of the same absolute value; and to subtract a negative quantity is the same in effect as to add a positive quantity of the same absolute value.

Cor. 2. Subtraction may be proved by adding the remainder to the subtrahend; the sum should be equal to the minuend.

Sch. 1. The results obtained by subtraction are easily explained by the diagram of Art. **68**. Thus,

(1) By the rule, if $+3$ is the subtrahend and $+8$ the minuend, the remainder is $+5$. This is as it should be, for, starting from the subtrahend 3, it is necessary to travel a distance of 5 in the *positive direction*, in order to reach the minuend 8.

(2) If $+8$ is the subtrahend and $+3$ the minuend, the remainder is -5. Starting from the subtrahend, it is necessary to travel a distance of 5 in the *negative direction*, in order to reach the minuend; hence the remainder *ought* to be -5.

(3) If -3 is the subtrahend and -8 the minuend, the remainder is -5. Starting from the subtrahend, it is necessary to travel a distance of 5 in the *negative direction*, in order to reach the minuend; hence the remainder *ought* to be -5.

(4) If -3 is the subtrahend and $+8$ the minuend, the remainder is 11. Starting from the subtrahend, it is necessary to travel a distance of 11 in the *positive direction*, in order to reach the minuend; hence the remainder *ought* to be $+11$.

(5) If $+8$ is the subtrahend and -3 the minuend, the remainder is -11. Starting from the subtrahend, it is necessary to travel a distance of 11 in the *negative direction*, in order to reach the minuend; hence the remainder *ought* to be -11.

Sch. 2. The beginner may have difficulty in understanding how one number can be subtracted from a smaller one, as in the second case of Sch. 1. In arithmetic this would be impossible; but subtraction in algebra has a wider meaning than in arithmetic.

Suppose that it is required to subtract 10 from 7. By the rule, the remainder is -3. This may be explained as follows:

It is evident that

$$7 = 10 - 3;$$

hence, if 10 be taken from $10 - 3$, the result will be the same as if 10 had been taken from 7. Removing the 10 from $10 - 3$, the result is -3.

The same result may be reached as follows:

Suppose B wishes to borrow \$10 from A, and that A has only \$7. To accommodate B, it will be necessary for A to borrow \$3. Then, when B receives \$10, A not only has no money, but is in debt \$3, and this debt of \$3 is indicated by — \$3.

SCH. 3. In applying the rule, the sign of every term of the subtrahend may be *actually* changed, but it is more convenient merely to *conceive* such change to be made.

EXERCISES.

Subtract

1. $a - b$ from $3a + b$. *Ans.* $2a + 2b$.
2. $2a - 3b$ from $3a - 5b$. *Ans.* $a - 2b$.
3. $3a - 5b + c$ from $4a - 2b - 3c$. *Ans.* $a + 3b - 4c$.
4. $2a - 5b + 6c$ from $a - 7b + 9c$. *Ans.* $-a - 2b + 3c$.
5. $2ax - 5by + 6cz$ from $18ax - 2by + 9cz$. *Ans.* $16ax + 3by + 3cz$.
6. $7ab^3 - 3a^2b^2 + 2ab^3$ from $10ab^3 - 5a^2b^2 + 4ab^3$. *Ans.* $3ab^3 - 2a^2b^2 + 2ab^3$.
7. $3ax^3 - 4bx^2y + 5cxy^2 - 7dy^3$ from $2ax^3 + 12bx^2y - 16cxy^2 + 18dy^3$. *Ans.* $-ax^3 + 16bx^2y - 21cxy^2 + 25dy^3$.
8. $5x^3 - 7y^3$ from $13y^3$. *Ans.* $-5x^3 + 20y^3$.
9. $7x - 3y + 5$ from 2. *Ans.* $-7x + 3y - 3$.
10. $3a - 2b + 2c - e$ from $2a + 3b - c + d$. *Ans.* $-a + 5b - 3c + d + e$.
11. $2a^2 - 5b^2 - 7xy - 5z^2$ from $3a^2 + 2b^2 - xy + z^2$. *Ans.* $a^2 + 7b^2 + 6xy + 6z^2$.
12. $c + d + e - f - g$ from $a - b$. *Ans.* $a - b - c - d - e + f + g$.

SYMBOLS OF AGGREGATION PRECEDED BY THE SIGN OF SUBTRACTION.

75. TH. *If a symbol of aggregation preceded by the sign — occurs in an expression, the value of the expression is not changed by omitting that symbol, provided the sign of every term embraced by it be changed.* Thus,

$$a-(b+c-d)=a-b-c+d \quad (\mathbf{74}).$$

COR. *The value of an expression is not changed by introducing a symbol of aggregation preceded by the sign —, provided the sign of every term embraced by it be changed.* Thus,

$$\begin{aligned} a-b-c-d &= a-b-(c+d)=a-(b+c+d) \\ &= a-d-(b+c). \end{aligned}$$

SCH. 1. The most careful attention should be given to the principles relating to the use of symbols of aggregation preceded by the sign — ; carelessness in regard to them is the most frequent cause of mistakes with beginners in Algebra.

SCH. 2. Sometimes one symbol of aggregation includes another, the inner one another, and so on, as in the expression

$$a-\{b-[c-(d-\overline{e-f})]\}.$$

In removing the symbols of aggregation from such an expression, any order may be pursued; but the beginner will be less liable to make mistakes, if he removes first that symbol which embraces simple terms only, then the symbol which embraced the one removed, and so on. For instance, he should treat the expression given above as follows:

$$\begin{aligned} a-\{b-[c-(d-\overline{e-f})]\} &= a-\{b-[c-(d-e+f)]\} \\ &= a-\{b-[c-d+e-f]\} \\ &= a-\{b-c+d-e+f\} \\ &= a-b+c-d+e-f. \end{aligned}$$

EXERCISES.

Remove the symbols of aggregation from each of the following expressions, and reduce each result to its simplest form:

1. $a - [b - (c + d)]$. *Ans.* $a - b + c + d$.
2. $a - [-(-b - c) + d]$. *Ans.* $a - b - c - d$.
3. $2b - 5a - [3b - (6a - 6b)]$. *Ans.* $a - 7b$.
4. $10x - \{5a - [4b + 2c - (6b - 5c + 2d)]\}$.
   Ans. $10x - 5a - 2b + 7c - 2d$.
5. $3a - a - b - [a + b + c - (a + b + c + d)]$.
   Ans. $2a - b + d$.
6. $2a - \{3b - [4a - (5b - 6a)]\}$. *Ans.* $12a - 8b$.
7. $a - \{a - [a - (a + x)]\}$. *Ans.* $-x$.
8. $4a - \{a - [a - (a - 3) + 3] - 3\}$
   $- \{-a - [-a - (-a + 3) + 3] - 3\}$.
   Ans. $4a + 12$.

A Short Method of Removing Symbols of Aggregation.

76. By considering the equation

$$a - \{b - [c - (d - \overline{e - f})]\} = a - b + c - d + e - f,$$

it will be seen that

The effect of an even number of negative signs is the same as that of one positive sign, and the effect of an odd number of negative signs is the same as that of one negative sign.

Thus, c in the first member is affected by two negative signs, and the corresponding term in the second member is $+c$; and d in the first member is affected by three negative signs, and the corresponding term in the second member is $-d$.

By observing this principle, the operation of removing symbols of aggregation may be very much abridged. Thus, in the expression

$$10x - \{6a - [3b + 2c - (6b - \overline{5c - 2d})]\},$$

$6a$ is affected by one negative sign, $3b$, by two, $2c$, by two, $6b$, by three, $5c$, by four, and $2d$, by five;

$$\begin{aligned} \therefore\ 10x - \{6a - [3b + 2c - (6b - \overline{5c - 2d})]\} &= 10x - 6a + 3b + 2c - 6b + 5c - 2d \\ &= 10x - 6a - 3b + 7c - 2d. \end{aligned}$$

EXERCISES.

Simplify

1. $a + [b - (c - d)]$. *Ans.* $a + b - c + d$.
2. $a - [-(b + c) - d]$. *Ans.* $a + b + c + d$.
3. $6a - 6b - [5a - (2b - \overline{a - b})]$. *Ans.* $-3b$.
4. $4x - \{6a - [2b - 3c - (4b - 2c - \overline{a - b})]\}$.
Ans. $4x - 5a - 3b - c$.
5. $5a - (a - b) - [-(a + b + c) - (a - b + 2c)]$.
Ans. $6a + b + 3c$.
6. $7a - \{4b - [3a - (-6a - 5b)]\}$. *Ans.* $16a + b$.
7. $-a - \{-a - [a + (-a - x)]\}$. *Ans.* $-x$.
8. $5x - \{x - [x - (x - 2) + 2] - 2\}$
$- \{-x - [x - (2 + x) + 2] - 2\}$.
Ans. $5x + 8$.

MULTIPLICATION.

THE PRODUCT OF MONOMIALS.

77. TH. *The product of two monomials is positive or negative, according as they have like signs or unlike signs.*

In the multiplication of one monomial by another there are four cases to be considered:

1st. When the multiplicand and multiplier are positive.

2d. When the multiplicand is negative and the multiplier positive.

3d. When the multiplicand is positive and the multiplier negative.

4th. When the multiplicand and multiplier are negative.

I. 1. Let it be required to multiply $+8$ by $+3$.

The multiplier $+3$ being 3 times $+1$, the product must be 3 times the multiplicand (**14**); but $+8+8+8=+24$;

$$\therefore\quad (+8)\times(+3)=+24.$$

2. Let it be required to multiply $+8$ by $+\frac{3}{5}$.

The multiplier $+\frac{3}{5}$ being 3 times $\frac{1}{5}$ of $+1$, the product must be 3 times $\frac{1}{5}$ of the multiplicand; but

$$+\frac{8}{5}+\frac{8}{5}+\frac{8}{5}=+\frac{24}{5};$$

$$\therefore\quad (+8)\times\left(+\frac{3}{5}\right)=+\frac{24}{5}.$$

3. Let it be required to multiply $+a$ by $+b$.

The reasoning used in the first example is applicable in any case in which the multiplier is a positive integer, and that used in the second example is applicable in any case in which the multiplier is a positive fraction;

$$\therefore\quad (+a)\times(+b)=+ab,$$

whether b is an integer or a fraction.

II. 1. Let it be required to multiply -8 by $+3$.

The multiplier being 3 times $+1$, the product must be 3 times the multiplicand; but $-8-8-8=-24$;

$$\therefore\quad (-8)\times(+3)=-24.$$

2. Let it be required to multiply -8 by $+\frac{3}{5}$.

The multiplier being 3 times $\frac{1}{5}$ of $+1$, the product must be 3 times $\frac{1}{5}$ of the multiplicand; but

$$-\frac{8}{5}-\frac{8}{5}-\frac{8}{5}=-\frac{24}{5};$$

$$\therefore\quad (-8)\times\left(+\frac{3}{5}\right)=-\frac{24}{5}.$$

3. Let it be required to multiply $-a$ by $+b$.

From the first and second examples of this case, it follows that

$$(-a)\times(+b)=-ab.$$

III. 1. Let it be required to multiply $+8$ by -3.

The multiplier being 3 times the *negative* of $+1$, the product must be 3 times the negative of the multiplicand; but the negative of the multiplicand is -8, and

$$-8-8-8=-24;$$

$$\therefore\quad (+8)\times(-3)=-24.$$

2. Let it be required to multiply $+8$ by $-\frac{3}{5}$.

The multiplier being 3 times $\frac{1}{5}$ of the negative of $+1$, the product must be 3 times $\frac{1}{5}$ of the negative of the multiplicand; but $\frac{1}{5}$ of the negative of the multiplicand is $-\frac{8}{5}$,

and

$$-\frac{8}{5}-\frac{8}{5}-\frac{8}{5}=-\frac{24}{5};$$

$$\therefore\quad (+8)\times\left(-\frac{3}{5}\right)=-\frac{24}{5}.$$

3. Let it be required to multiply $+a$ by $-b$.

From the first and second examples of this case, it follows that

$$(+a) \times (-b) = -ab.$$

IV. 1. Let it be required to multiply -8 by -3.

The multiplier being 3 times the negative of $+1$, the product must be 3 times the negative of the multiplicand; but the negative of the multiplicand is $+8$, and

$$+8+8+8=+24;$$
$$\therefore\ (-8)\times(-3)=+24.$$

2. Let it be required to multiply -8 by $-\frac{3}{5}$.

The multiplier being 3 times $\frac{1}{5}$ of the negative of $+1$, the product must be 3 times $\frac{1}{5}$ of the negative of the multiplicand; but $\frac{1}{5}$ of the negative of the multiplicand is $+\frac{8}{5}$,

and $$+\frac{8}{5}+\frac{8}{5}+\frac{8}{5}=+\frac{24}{5};$$
$$\therefore\ (-8)\times\left(-\frac{3}{5}\right)=+\frac{24}{5}.$$

3. Let it be required to multiply $-a$ by $-b$.

From the first and second examples of this case, it follows that

$$(-a)\times(-b)=+ab.$$

Cor. 1. If the multiplier is positive, the product has the same sign as the multiplicand; but if the multiplier is negative, the sign of the product is the opposite of that of the multiplicand. Thus,

$$(+7)\times(+5)=+35, \qquad (-7)\times(+5)=-35,$$
$$(+7)\times(-5)=-35, \qquad (-7)\times(-5)=+35.$$

COR. 2. If the number of negative factors which enter into a product is even, the product is positive; but if the number of negative factors is odd, the product is negative. Thus,

$$(+7) \times (+5) \times (-3) \times (-8) = +840,$$

and $$(+7) \times (-5) \times (-3) \times (-8) = -840.$$

COR. 3. The product of two or more factors is the same, in whatever order they may be taken. For a change in the order of the factors effects no change in the *absolute* value of the product (National Arithmetic, Art. **63**); and since the sign of the product depends only on the number of negative factors, a change in their order effects no change in the *sign* of the product. Thus,

$$\begin{aligned}(+2)\times(+3)\times(-4)\times(-5) &= (+2)\times(-4)\times(+3)\times(-5)\\ &= (+2)\times(-4)\times(-5)\times(+3).\end{aligned}$$

78. To find the product of monomials.

Let it be required to find the product of $-7a^3b^2$, $5a^2b^4c$, and $3b^2d^3$.

$$\begin{aligned}-7a^3b^2 \times 5a^2b^4c \times 3b^2d^3 &= -7 \times 5 \times 3 \times a^3a^2b^2b^4b^2cd^3\\ &= -105a^3a^2b^2b^4b^2cd^3 \text{ (77, Cor. 3).}\end{aligned}$$

Here a occurs as a factor 5 times, b, 8 times, c, once, and d, 3 times; hence the product may be written thus, $-105a^5b^8cd^3$.

RULE.

On the right of the product of the given coefficients write all the letters of the given factors, and give to each letter that is common to two or more of the given factors an exponent equal to the sum of its exponents in those factors.

COR. The degree of the product of monomials is equal to the sum of the degrees of the factors, since all the factors of each monomial appear in the product.

Sch. Literal factors are usually written in alphabetical order.

EXERCISES.

Find the product of

1. a^3b and a^2b^2. *Ans.* a^5b^3.
2. $-3a$ and $2bc$. *Ans.* $-6abc$.
3. $5a^2bc$ and $-7a^3b^2c$. *Ans.* $-35a^5b^3c^2$.
4. a^2b, c^3d^2, x, $-2z$. *Ans.* $-2a^2bc^3d^2xz$.
5. $3a^mb^n$, $5a^nb^m$, $-7c^md^n$. *Ans.* $-105a^{m+n}b^{m+n}c^md^n$.
6. $2a^x$, $-3b^y$, $5c^z$, $-abc$. *Ans.* $30a^{x+1}b^{y+1}c^{z+1}$.

THE PRODUCT OF A POLYNOMIAL AND A MONOMIAL.

79. To multiply a polynomial by a monomial.

Let it be required to multiply $a + b - c$ by d.

The partial products are ad, $+ bd$, and $-cd$ (77); hence the complete product is $ad + bd - cd$.

$$\begin{array}{l} a + b - c \\ d \\ \hline ad + bd - cd. \end{array}$$

RULE.

Multiply each term of the polynomial by the monomial, and find the sum of the partial products.

Cor. If a homogeneous polynomial be multiplied by a monomial, the product will be homogeneous.

EXERCISES.

Multiply

1. $3ax - 2ay + z$ by $4a$. *Ans.* $12a^2x - 8a^2y + 4az$.
2. $5x^3 - 2x^2 + x - 4$ by $-15ax$.
   Ans. $-75ax^4 + 30ax^3 - 15ax^2 + 60ax$.
3. $-ax + by - 3cz^2$ by $-2e^2x^2$.
   Ans. $2ae^2x^3 - 2be^2x^2y + 6ce^2x^2z^2$.

4. $x^2 - px + q$ by px. *Ans.* $px^3 - p^2x^2 + pqx$.

5. $7x^2 - 2x + 4a$ by $3a^2$. *Ans.* $21a^2x^2 - 6a^2x + 12a^3$.

6. $4x^2y + 3x - 2y$ by $-3xy$.
Ans. $-12x^3y^2 - 9x^2y + 6xy^2$.

7. $a^2b + cd - ef$ by ax. *Ans.* $a^3bx + acdx - aefx$.

8. $a^2 + 2ab + b^2$ by $-bc$.
Ans. $-a^2bc - 2ab^2c - b^3c$.

9. $-ab - cd - ef$ by $-g$. *Ans.* $abg + cdg + efg$.

10. $a^2x + b^2y - c^2z$ by $-m^2n$.
Ans. $-a^2m^2nx - b^2m^2ny + c^2m^2nz$.

11. $a^4 + a^3x - a^2x^2 + ax^3 - x^4$ by $-(-x^2)$.
Ans. $a^4x^2 + a^3x^3 - a^2x^4 + ax^5 - x^6$.

12. $\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{4}$ by $\frac{2}{3}x^2$. *Ans.* $\frac{1}{3}x^4 - \frac{2}{9}x^3 + \frac{1}{6}x^2$.

THE PRODUCT OF POLYNOMIALS.

80. To multiply one polynomial by another.

$$\begin{array}{l} a - b \\ \underline{c - d \qquad\qquad\qquad} \\ ac - bc - ad + bd. \end{array}$$

Let it be required to multiply $a - b$ by $c - d$.

$$(a - b)\,c = ac - bc,$$

and $(a-b)\,(-d) = -ad + bd$ **(79)**;

hence the complete product is $ac - bc - ad + bd$.

RULE.

Multiply the multiplicand by each term of the multiplier in succession, and find the sum of the partial products.

Cor. 1. The product of three polynomials may be found by multiplying the product of two of them by the third; the product of four polynomials may be found by multiplying the product of three of them by the fourth, or by multiplying the product of two of them by the product of the other two; and so on.

COR. 2. If two polynomials are homogeneous, their product will be homogeneous.

EXERCISES.

$$\begin{array}{lllllll}
\text{1. Multiply} & 4a^2 & - 5ab & + 6b^2 & & \\
\text{by} & 2a^2 & + 3ab & - 4b^2 & & \\
\cline{2-4}
 & 8a^4 & - 10a^3b & + 12a^2b^2 & & \\
 & & + 12a^3b & - 15a^2b^2 & + 18ab^3 & \\
 & & & - 16a^2b^2 & + 20ab^3 & - 24b^4 \\
\cline{2-6}
\text{Product,} & 8a^4 & + 2a^3b & - 19a^2b^2 & + 38ab^3 & - 24b^4.
\end{array}$$

$$\begin{array}{llllll}
\text{2. Multiply} & 2a^2 & + 3a & + 4 & & \\
\text{by} & 2a^2 & - 3a & + 4 & & \\
\cline{2-4}
 & 4a^4 & + 6a^3 & + 8a^2 & & \\
 & & - 6a^3 & - 9a^2 & - 12a & \\
 & & & + 8a^2 & + 12a & + 16 \\
\cline{2-6}
\text{Product,} & 4a^4 & & + 7a^2 & & + 16.
\end{array}$$

$$\begin{array}{ll}
\text{3. Multiply} & a^2 - 5b^2 + 2c^2 \\
\text{by} & a^2 - b^3 \\
\cline{2-2}
\text{Product,} & a^4 - 5a^2b^2 + 2a^2c^2 - a^2b^3 + 5b^5 - 2b^3c^2.
\end{array}$$

$$\begin{array}{llllll}
\text{4. Multiply} & 2x^3 & + 4x^2 & + 8x & + 16 & \\
\text{by} & 3x & - 6 & & & \\
\cline{2-5}
 & 6x^4 & + 12x^3 & + 24x^2 & + 48x & \\
 & & - 12x^3 & - 24x^2 & - 48x & - 96 \\
\cline{2-6}
\text{Product,} & 6x^4 & & & & - 96.
\end{array}$$

5. Multiply $5x + 4y$ by $2a + 3b$.
Ans. $10ax + 8ay + 15bx + 12by$.

6. Multiply $x + y$ by $x - y$. *Ans.* $x^2 - y^2$.

7. Multiply $3x^2 + 2x$ by $4x - 7$.
Ans. $12x^3 - 13x^2 - 14x$.

8. Multiply $a^2 + ab + b^2$ by $a - b$. *Ans.* $a^3 - b^3$.

9. Multiply $3x^2 - 2x + 5$ by $2x^2 - 6x + 7$.

Ans. $6x^4 - 22x^3 + 43x^2 - 44x + 35$.

10. Multiply $14ac - 3ab + 2$ by $ac - ab + 1$.

Ans. $14a^2c^2 - 17a^2bc + 16ac + 3a^2b^2 - 5ab + 2$.

11. Multiply $8ab - 3b + c$ by $2ab + b - 5c$.

Ans. $16a^2b^2 + 2ab^2 - 38abc - 3b^2 + 16bc - 5c^2$.

12. Multiply $7a^2b - 5ab^2 + 6b^3$ by $2a^3 - 3a^2b - 5b^3$.

Ans. $14a^5b - 31a^4b^2 + 27a^3b^3 - 53a^2b^4 + 25ab^5 - 30b^6$.

13. Multiply $x^4 + x^3 + x^2 + x + 1$ by $x - 1$.

Ans. $x^5 - 1$.

14. Multiply $x^6 + x^4 + x^2 + 1$ by $x^6 - x^4 + x^2 - 1$.

Ans. $x^{12} + x^8 - x^4 - 1$.

15. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.

Ans. $a^4 + a^2b^2 + b^4$.

16. Multiply $x^3 - 3x^2 + 2x + 1$ by $2x^2 - 5x + 4$.

Ans. $2x^5 - 11x^4 + 23x^3 - 20x^2 + 3x + 4$.

17. Multiply $5a^2 - 3ab + 7b^2$ by $3a - b$.

Ans. $15a^3 - 14a^2b + 24ab^2 - 7b^3$.

18. Multiply $a^2 + 2ab + b^2 - c^2$ by $a^2 + 2ab - b^2 + c^2$.

Ans. $a^4 + 4a^3b + 4a^2b^2 + 2b^2c^2 - b^4 - c^4$.

19. Multiply $(a - 2)^2$ by $(a + 2)^2$.

Ans. $a^4 - 8a^2 + 16$.

20. Multiply $(a^2 + ab + b^2)^2$ by $(a - b)^2$.

Ans. $a^6 - 2a^3b^3 + b^6$.

21. Find the product of $x - 1$, $x + 2$, and $x + 3$.

Ans. $x^3 + 4x^2 + x - 6$.

22. Find the product of $x + a$, $x + b$, and $x + c$.

Ans. $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$.

23. Find the product of $x + a$, $x - b$, and $x - c$.

Ans. $x^3 + (a - b - c)x^2 + (bc - ab - ac)x + abc$.

24. Show that $(x + 2)^3 = x^3 + 6x^2 + 12x + 8$.

25. Show that $(a + b + c)^3 = a^3 + b^3 + c^3 + 3ab^2 + 3a^2b + 3ac^2 + 3a^2c + 3bc^2 + 3b^2c + 6abc$.

26. Show that $(a+b)^2-(a-b)^2=4ab$.

27. Show that $4ab(a^2+b^2)=(a^2+ab+b^2)^2-(a^2-ab+b^2)^2$.

28. Show that $(m-n)(m+n)(m^2+n^2)(m^4+n^4)=m^8-n^8$.

29. Simplify

$$(a+b)(b+c)-[(c+d)(a+d)+(a+c)(b-d)]$$

Ans. b^2-d^2.

30. Simplify $(a+b+c)^2-[a(b+c-a)+b(a+c-b)+c(a+b-c)]$. *Ans.* $2(a^2+b^2+c^2)$.

31. Simplify $(a+b+c+d)^2+(a-b-c+d)^2+(a-b+c-d)^2+(a+b-c-d)^2$.

Ans. $4(a^2+b^2+c^2+d^2)$.

81. TH. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.*

Let a and b represent any two quantities; then $a+b$ will represent their sum. It is required to show that

$$(a+b)^2=a^2+2ab+b^2.$$

The product obtained by multiplying $a+b$ by $a+b$ is $a^2+2ab+b^2$;

$$\begin{array}{lll} a+b & & \\ \underline{a+b} & & \\ a^2+ab & & \\ \underline{\quad\;\; +ab+b^2} & & \\ a^2+2ab+b^2. & & \end{array}$$

$$\therefore\quad (a+b)^2=a^2+2ab+b^2.$$

SCH. By means of this theorem, the process of squaring the sum of two quantities may be very much abridged. For example,

Let it be required to square $2m+3n$.

The square of $2m$ is $4m^2$;
twice the product of $2m$ and $3n$ is $12mn$;
and the square of $3n$ is $9n^2$;
hence, by the theorem,

$$(2m+3n)^2=4m^2+12mn+9n^2.$$

EXERCISES.

Write, by means of the theorem, the squares of

1. $5 + 2$.
2. $ax + by$.
3. $2a^2bx + 3c^2d$.
4. $c^2 + 2d$.
5. $x^3 + 2$.
6. $(a+b)+(c+d)$.
7. $4a + 3b$.
8. $a + (-b)$.
9. $(a+b)+(-c)$.

82. Th. *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

$$\begin{array}{lll} a & - b & \\ a & - b & \\ \hline a^2 & - ab & \\ & - ab & + b^2 \\ \hline a^2 & - 2ab & + b^2. \end{array}$$

Let a and b represent any two quantities; then $a-b$ will represent their difference. It is required to show that

$$(a - b)^2 = a^2 - 2ab + b^2.$$

The product obtained by multiplying $a-b$ by $a-b$ is $a^2-2ab+b^2$;

$$\therefore \quad (a - b)^2 = a^2 - 2ab + b^2.$$

EXERCISES.

Write, by means of the theorem, the squares of

1. $5 - 2$.
2. $ax - by$.
3. $c^2 - 2d$.
4. $x^3 - 2$.
5. $4a - 3b$.
6. $a - (-b)$.

83. Th. *The product of the sum and the difference of two quantities is equal to the difference of their squares.*

$$\begin{array}{lll} a & + b & \\ a & - b & \\ \hline a^2 & + ab & \\ & - ab & - b^2 \\ \hline a^2 & & - b^2. \end{array}$$

Let a and b represent any two quantities; then $a + b$ will represent their sum, and $a - b$, their difference. It is required to show that

$$(a + b)(a - b) = a^2 - b^2.$$

The product obtained by multiplying $a + b$ by $a - b$ is $a^2 - b^2$;

$$\therefore \quad (a + b)(a - b) = a^2 - b^2.$$

EXERCISES.

Write, by means of the theorem, the products of

1. $5 + 2$ and $5 - 2$.
2. $3a + 2b$ and $3a - 2b$.
3. $x + 1$ and $x - 1$.
4. $ax^2 + bx$ and $ax^2 - bx$.
5. $a^2 + b^2$ and $a^2 - b^2$.
6. $(a+b)+c$ and $(a+b)-c$.

DIVISION.

THE QUOTIENT OF TWO MONOMIALS.

34. Th. *The quotient obtained by dividing one monomial by another is positive or negative, according as they have like signs or unlike signs.*

The quotient obtained by dividing one monomial by another must have such a sign as will make the sign of the product of the divisor and quotient like that of the dividend (**16**). Hence, if the dividend and divisor have *like* signs, the sign of the quotient must be +; and, if the dividend and divisor have *unlike* signs, the sign of the quotient must be —. Thus,

$$\frac{+ab}{+a} = +b, \text{ for } +a \times (+b) = +ab,$$

$$\frac{-ab}{-a} = +b, \text{ for } -a \times (+b) = -ab,$$

$$\frac{+ab}{-a} = -b, \text{ for } -a \times (-b) = +ab,$$

$$\frac{-ab}{+a} = -b, \text{ for } +a \times (-b) = -ab.$$

Cor. If the dividend is positive, the quotient has the same sign as the divisor; but if the dividend is negative, the sign of the quotient is the opposite of that of the divisor.

85. To find the quotient of two monomials.

Let it be required to divide $35a^5b^4c^3d^2e$ by $-7a^3b^2c$.

(1.) Since the product of the divisor and quotient must be $35a^5b^4c^3d^2e$ (**16**), the coefficient of the quotient must be such a number that, if it be multiplied by -7, the coefficient of the divisor, the product will be 35; hence the coefficient of the quotient is -5, which is obtained by dividing 35 by -7.

(2.) If a letter is common to the divisor and quotient, the exponent of that letter in the divisor added to its exponent in the quotient will give its exponent in the dividend (**78**); hence, if a letter is common to the dividend and divisor, its exponent in the quotient is found by subtracting its exponent in the divisor from that in the dividend.

(3.) The letters d and e occur in the dividend, but not in the divisor, and as the product of the divisor and quotient is equal to the dividend, these letters must occur in the quotient with the same exponents as in the dividend.

$$\therefore \quad \frac{35a^5b^4c^3d^2e}{-7a^3b^2c} = -5a^2b^2c^2d^2e.$$

RULE.

1. *Divide the coefficient of the dividend by that of the divisor. The result will be the coefficient of the required quotient.*

2. *On the right of the coefficient of the quotient write all the letters of the dividend.*

3. *Give to each letter that is common to the dividend and divisor an exponent which is found by subtracting its exponent in the divisor from that in the dividend, and to each letter that is not common the exponent which it has in the dividend.*

COR. $$\frac{a^m}{a^m} = a^{m-m} = a^0, \quad \text{and} \quad \frac{a^m}{a^m} = 1;$$

$$\therefore \quad a^0 = 1.$$

Hence, if the exponent of a letter is the same in the dividend as in the divisor, that letter may be dropped from the quotient; and any letter with 0 for its exponent may be introduced as a factor into any expression. Thus,

$$5a^2b^0c^2d = 5a^2c^2d = 5a^2c^2de^0.$$

EXERCISES.

Divide

1. $24a^5b^3c^2$ by $6a^2b$. *Ans.* $4a^3b^2c^2$.
2. $6a^2xy$ by $-3ax$. *Ans.* $-2ay$.
3. $-12a^3b^2c^3mn$ by $3bc^2n$. *Ans.* $-4a^3bcm$.
4. $-15a^6x^4$ by $-3a^4x^3$. *Ans.* $5a^2x$.
5. $-81a^{m+n}x^{m+n}$ by $-3a^nx^m$. *Ans.* $27a^mx^n$.
6. $-27a^mx^n$ by $-3a^nx^m$. *Ans.* $9a^{m-n}x^{n-m}$.

86. EXACT DIVISION. The quotient of two monomials is said to be *entire*, or *integral*, when its coefficient is integral, and when the exponent of each letter in it is *positive*. If the quotient is entire, the division is said to be *exact*.

Hence, in order that the division may be exact,

1st. The coefficient of the dividend must be exactly divisible by that of the divisor.

2d. The dividend must contain all the letters that occur in the divisor.

3d. The exponent of each letter in the dividend must be equal to, or greater than, the exponent of that letter in the divisor.

THE QUOTIENT OF A POLYNOMIAL BY A MONOMIAL.

87. To find the quotient of a polynomial by a monomial.

Let it be required to divide $ab - bc + bd$ by b.

$$\begin{array}{r|l} b\,) & \underline{ab - bc + bd} \\ & a - \;\; c + \;\; d. \end{array}$$

$$\frac{ab - bc + bd}{b} = a - c + d;$$

for $$(a - c + d)\,b = ab - bc + bd.$$

The same result may be obtained by dividing each term of the dividend by the divisor, and finding the sum of the partial quotients a, $-c$, and $+d$.

RULE.

Divide each term of the polynomial by the monomial, and find the sum of the partial quotients.

EXERCISES.

Divide

1. $6ab - 9ax + 3a^2$ by $3a$. *Ans.* $2b - 3x + a$.
2. $-25a^2b + 15ab^2 + 10ab$ by $-5ab$. *Ans.* $5a - 3b - 2$.
3. $-3bcd - 12bcx + 9b^2c$ by $3bc$. *Ans.* $-d - 4x + 3b$.
4. $3ax^3 + 6x^2 + 3ax - 15x$ by $3x$. *Ans.* $ax^2 + 2x + a - 5$.
5. $-25a^2bx + 15a^2cx^2 - 5a^3bcx^2$ by $5a^2x$. *Ans.* $-5b + 3cx - abcx$.
6. $-12a^4b^6 + 16a^5b^5 - 20a^6b^4 + 28a^7b^3$ by $-4a^4b^3$. *Ans.* $3b^3 - 4ab^2 + 5a^2b - 7a^3$.

THE QUOTIENT OF TWO POLYNOMIALS.

88. To find the quotient of two polynomials.

Let it be required to divide $26a^2b^2 + 24ab^3 - 48a^3b + 10a^4$ by $2a^2 - 8ab$.

The operation may be conveniently arranged as follows:

$$\begin{array}{rr|l}
 & \text{Dividend.} & \text{Divisor.} \\
 & 10a^4 - 48a^3b + 26a^2b^2 + 24ab^3 & 2a^2 - 8ab \\ \cline{3-3}
 & \underline{10a^4 - 40a^3b \qquad\qquad\qquad\quad} & 5a^2 - 4ab - 3b^2. \\
\text{1st Rem.} = & -\ 8a^3b + 26a^2b^2 + 24ab^3 & \\
 & \underline{-\ 8a^3b + 32a^2b^2 \qquad\quad} & \\
\text{2d Rem.} = & -\ 6a^2b^2 + 24ab^3 & \\
 & \underline{-\ 6a^2b^2 + 24ab^3} &
\end{array}$$

The term of the dividend which contains the highest power of a must be equal to the product obtained by multiplying the term of the divisor which contains the highest power of a by the term of the quotient which contains the highest power of the same letter **(16)**. Therefore, if the dividend and divisor are arranged according to the descending powers of a, the first term of the quotient may be found by *dividing* the first term of the dividend by the first term of the divisor. Hence, the first term of the quotient is $5a^2$.

Again, the dividend is equal to the sum of the partial products obtained by multiplying the divisor by each term of the quotient in succession; therefore, if the product obtained by multiplying the divisor by $5a^2$ be subtracted from the dividend, the remainder, $-8a^3b + 26a^2b^2 + 24ab^3$, will be equal to the sum of the partial products obtained by multiplying the divisor by the remaining terms of the quotient; hence this remainder may be used as a new dividend to obtain the second term of the quotient. Proceeding in this manner, the complete quotient is found to be $5a^2 - 4ab - 3b^2$.

A similar course of reasoning is applicable when the dividend and divisor are arranged according to the *ascending* powers of a common letter.

RULE.

1. *Arrange the dividend and divisor according to the powers of some common letter.*

2. *Divide the first term of the dividend by the first term of the divisor. The result will be the first term of the quotient.*

3. *Multiply the divisor by this term, and subtract the product from the dividend.*

4. *Divide the first term of the remainder by the first term of the divisor. The result will be the second term of the quotient.*

5. *Multiply the divisor by this term and subtract the product from the remainder.*

6. *Continue this operation until a remainder is found equal to* 0, *or whose first term is not exactly divisible by the divisor.*

COR. In order that the division of one polynomial by another may be exact:

1st. The first term of the arranged dividend must be exactly divisible by the first term of the arranged divisor.

2d. The last term of the arranged dividend must be exactly divisible by the last term of the arranged divisor.

3d. The first term of any arranged remainder must be exactly divisible by the first term of the arranged divisor.

SCH. The situation of the divisor in regard to the dividend is a matter of arbitrary arrangement; but by placing it on the right, it is more easily multiplied by the several terms of the quotient as they are found.

EXERCISES.

1. Divide $2x^2 - x - 1$ by $2x + 1$.

$$\begin{array}{l|l} 2x^2 - x - 1 & \underline{2x + 1} \\ \underline{2x^2 + x \quad\quad} & x - 1. \\ \quad - 2x - 1 & \\ \quad \underline{- 2x - 1} & \end{array}$$

2. Divide $2x^2 - xy - y^2$ by $2x + y$.

$$\begin{array}{l|l} 2x^2 - xy - y^2 & \underline{2x + y} \\ \underline{2x^2 + xy \quad\quad} & x - y. \\ \quad - 2xy - y^2 & \\ \quad \underline{- 2xy - y^2} & \end{array}$$

3. Divide $1 - x^3$ by $1 - x$.

$$\begin{array}{l|l} 1 - x^3 & \underline{1 - x \quad\quad\quad} \\ \underline{1 - x \quad\quad} & 1 + x + x^2. \\ \quad x - x^3 & \\ \quad \underline{x - x^2 \quad} & \\ \quad\quad x^2 - x^3 & \\ \quad\quad \underline{x^2 - x^3} & \end{array}$$

4. Divide $x^4 - y^4$ by $x + y$.

$$\begin{array}{l|l} x^4 - y^4 & \underline{x + y \quad\quad\quad\quad\quad\quad} \\ \underline{x^4 + x^3y \quad\quad} & x^3 - x^2y + xy^2 - y^3. \\ \quad - x^3y - y^4 & \\ \quad \underline{- x^3y - x^2y^2 \quad} & \\ \quad\quad x^2y^2 - y^4 & \\ \quad\quad \underline{x^2y^2 + xy^3 \quad} & \\ \quad\quad\quad - xy^3 - y^4 & \\ \quad\quad\quad \underline{- xy^3 - y^4} & \end{array}$$

5. Divide $12a^2 + 11ab - 8ac - 15b^2 + 6bc$ by $3a + 5b - 2c$.

$$
\begin{array}{l|l}
12a^2 + 11ab - 8ac - 15b^2 + 6bc & 3a + 5b - 2c \\ \cline{2-2}
12a^2 + 20ab - 8ac & 4a - 3b. \\ \cline{1-1}
\qquad - 9ab \qquad\qquad - 15b^2 + 6bc & \\
\qquad - 9ab \qquad\qquad - 15b^2 + 6bc & \\ \cline{1-1}
\end{array}
$$

6. Divide $x^3 + y^3$ by $x - y$.

$$
\begin{array}{rr|l}
& x^3 + y^3 & x - y \\ \cline{3-3}
& x^3 - x^2y & x^2 + xy + y^2 + \dfrac{2y^3}{x - y}. \\ \cline{2-2}
& x^2y + y^3 & \\
& x^2y - xy^2 & \\ \cline{2-2}
& xy^2 + y^3 & \\
& xy^2 - y^3 & \\ \cline{2-2}
\text{Rem.} & 2y^3 &
\end{array}
$$

In this example, the third remainder, $2y^3$, is not exactly divisible by $x - y$. The division may be indicated thus, $\frac{2y^3}{x - y}$, and the quotient of $x^3 + y^3$ divided by $x - y$ must include the expression $\frac{2y^3}{x - y}$, to be complete.

Divide

7. $16a^2b^2 + 2ab^2 - 38abc - 3b^2 + 16bc - 5c^2$ by $2ab + b - 5c$. *Ans.* $8ab - 3b + c$.

8. $14a^5b - 31a^4b^2 + 27a^3b^3 - 53a^2b^4 + 25ab^5 - 30b^6$ by $2a^3 - 3a^2b - 5b^3$. *Ans.* $7a^2b - 5ab^2 + 6b^3$.

9. $x^5 - 1$ by $x - 1$. *Ans.* $x^4 + x^3 + x^2 + x + 1$.

10. $x^{12} + x^8 - x^4 - 1$ by $x^6 - x^4 + x^2 - 1$. *Ans.* $x^6 + x^4 + x^2 + 1$.

11. $a^4 + a^2b^2 + b^4$ by $a^2 - ab + b^2$. *Ans.* $a^2 + ab + b^2$.

12. $2x^5 - 11x^4 + 23x^3 - 20x^2 + 3x + 4$ by $2x^2 - 5x + 4$. *Ans.* $x^3 - 3x^2 + 2x + 1$.

13. $15a^3 - 14a^2b + 24ab^2 - 7b^3$ by $3a - b$.
Ans. $5a^2 - 3ab + 7b^2$.

14. $a^4 + 4a^3b + 4a^2b^2 + 2b^2c^2 - b^4 - c^4$ by $a^2 + 2ab - b^2 + c^2$.
Ans. $a^2 + 2ab + b^2 - c^2$.

15. $a^4 - 8a^2 + 16$ by $(a + 2)^2$. *Ans.* $a^2 - 4a + 4$.

16. $a^6 - 2a^3b^3 + b^6$ by $(a^2 + ab + b^2)^2$.
Ans. $a^2 - 2ab + b^2$.

17. $x^3 + 4x^2 + x - 6$ by $(x + 2)(x + 3)$.
Ans. $x - 1$.

18. $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$ by $(x + b)(x + c)$. *Ans.* $x + a$.

19. $x^3 + (a - b - c)x^2 + (bc - ab - ac)x + abc$ by $(x - b)(x - c)$. *Ans.* $x + a$.

20. $a^8 - 3a^7 - 9a^6 - 5a^3 + 13a^2 + 51a + 18$ by $a^2 - 3a - 9$.
Ans. $a^6 - 5a - 2$.

21. $x^5 + x^4 - x - 1$ by $x^3 + x^2 + x + 1$.
Ans. $x^2 - 1$.

22. $a^6 + 4a^5 + 4a^4 - 16a^2 - 64a - 64$ by $a^3 - 2a^2 + 4a - 8$.
Ans. $a^3 + 6a^2 + 12a + 8$.

23. $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$ by $1 - 2x + x^2$.
Ans. $1 - 3x + 3x^2 - x^3$.

24. $x^8 + x^4 + 1$ by $(x^2 + x + 1)(x^2 - x + 1)$.
Ans. $x^4 - x^2 + 1$.

25. $(a - 2)(a - 1)a(a + 1)(a + 2)$ by $a^2 - a - 2$.
Ans. $a^3 + a^2 - 2a$.

26. $(a^3 - 3a + 2)(a^3 - 3a^2 + 2)$ by $a^2 - 2a + 1$.
Ans. $a^4 - a^3 - 6a^2 + 2a + 4$.

27. $x^4 + 3x^3 + 2x + 15$ by $x - 3$.
Ans. $x^3 + 6x^2 + 18x + 56 + \dfrac{183}{x - 3}$.

28. $3x^6 + 2x^4 - 7$ by $x - 5$.
Ans. $3x^5 + 15x^4 + 77x^3 + 385x^2 + 1925x + 9625 + \dfrac{48118}{x - 5}$.

89. TH. *The difference between the like powers of any two quantities is exactly divisible by the difference between the quantities.*

Let x and a represent any two quantities, and let n be any positive integer; then $x^n - a^n$ is exactly divisible by $x - a$.

If $x^2 - a^2$, $x^3 - a^3$, and $x^4 - a^4$ be divided by $x - a$, it will be found that

$$\frac{x^2 - a^2}{x - a} = x + a = a^0x + ax^0 \text{ (**85**, Cor.)};$$

$$\frac{x^3 - a^3}{x - a} = x^2 + ax + a^2 = a^0x^2 + ax + a^2x^0;$$

$$\frac{x^4 - a^4}{x - a} = x^3 + ax^2 + a^2x + a^3 = a^0x^3 + ax^2 + a^2x + a^3x^0.$$

The division in each of these cases is *exact*, and it will be observed, that all the terms of the quotient are positive; that the number of terms in the quotient is equal to the exponent of x or of a in the dividend; that the first term of the quotient is x with an exponent one less than the exponent of x in the dividend, and in each of the other terms the exponent of x is one less than in the preceding term; that the last term is a with an exponent one less than the exponent of a in the dividend, and in each of the other terms the exponent of a is one less than in the following term.

Now, if these principles are *general*, the quotient obtained by dividing $x^n - a^n$ by $x - a$ *ought* to be

$$x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-3}x^2 + a^{n-2}x + a^{n-1}.$$

That this *is* the quotient may be proved by multiplying it by $x - a$, thus:

$$\begin{array}{l}
x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots\ldots + a^{n-3}x^2 + a^{n-2}x + a^{n-1} \\
x - a \\
\hline
x^n + ax^{n-1} + a^2x^{n-2} + \ldots\ldots + a^{n-3}x^3 + a^{n-2}x^2 + a^{n-1}x \\
\qquad - ax^{n-1} - a^2x^{n-2} - a^3x^{n-3} - \ldots\ldots - a^{n-2}x^2 - a^{n-1}x - a^n \\
\hline
x^n \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - a^n.
\end{array}$$

The product is $x^n - a^n$, which is equal to the dividend;

$$\therefore \frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots\ldots + a^{n-3}x^2 + a^{n-2}x + a^{n-1}.$$

The principles deduced from the division of $x^2 - a^2$, $x^3 - a^3$, and $x^4 - a^4$, by $x - a$ are, therefore, universally true.

EXERCISES.

Write the quotients of

1. $\dfrac{x^6 - a^6}{x - a}$.
2. $\dfrac{x^7 - a^7}{x - a}$.
3. $\dfrac{x^8 - a^8}{x - a}$.
4. $\dfrac{x^9 - a^9}{x - a}$.
5. $\dfrac{x^{10} - a^{10}}{x - a}$.
6. $\dfrac{x^{11} - a^{11}}{x - a}$.
7. $\dfrac{x^{12} - a^{12}}{x - a}$.
8. $\dfrac{a^4 - (-b)^4}{a - b}$.
9. $\dfrac{a^3 - 8}{a - 2}$.

90. TH. *The difference between the like even powers of two quantities is exactly divisible by the sum of the quantities.*

Let x and a represent any two quantities, and let n be an *even positive* integer; then $x^n - a^n$ is exactly divisible by $x + a$.

If n is *even*, $\dfrac{x^n - (-a)^n}{x - (-a)} = \dfrac{x^n - a^n}{x + a}$ (**77**, Cor. 2).

But $x^n - (-a)^n$ is exactly divisible by $x - (-a)$ (**89**); hence, $x^n - a^n$, which is equal to $x^n - (-a)^n$, must be exactly divisible by $x + a$, which is equal to $x - (-a)$.

The form of the quotient may be learned by finding a few of its terms by actual division, thus:

$$\begin{array}{l|l}
x^n - a^n & x + a \\ \cline{2-2}
\underline{x^n + ax^{n-1}} & x^{n-1} - ax^{n-2} + a^2x^{n-3} - a^3x^{n-4} + \text{etc.} \\
\quad - ax^{n-1} - a^n & \\
\quad \underline{- ax^{n-1} - a^2x^{n-2}} & \\
\qquad\qquad + a^2x^{n-2} - a^n & \\
\qquad\qquad \underline{+ a^2x^{n-2} + a^3x^{n-3}} & \\
\qquad\qquad\qquad\qquad - a^3x^{n-3} - a^n & \\
\qquad\qquad\qquad\qquad \underline{- a^3x^{n-3} - a^4x^{n-4}} & \\
\qquad\qquad\qquad\qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot &
\end{array}$$

$$\therefore \frac{x^n - a^n}{x + a} = x^{n-1} - ax^{n-2} + a^2x^{n-3} - a^3x^{n-4} + \ldots \\ - a^{n-3}x^2 + a^{n-2}x - a^{n-1},$$

when n is even.

SCH. The terms of the quotient are alternately positive and negative.

EXERCISES.

Write the quotients of

1. $\dfrac{x^2 - a^2}{x + a}$.
2. $\dfrac{x^4 - a^4}{x + a}$.
3. $\dfrac{x^6 - a^6}{x + a}$.
4. $\dfrac{x^2 - 1}{x + 1}$.
5. $\dfrac{x^4 - 1}{x + 1}$.
6. $\dfrac{x^6 - 1}{x + 1}$.
7. $\dfrac{a^8 - b^8}{a + b}$.
8. $\dfrac{a^{10} - b^{10}}{a + b}$.
9. $\dfrac{x^{12} - y^{12}}{x + y}$.

91. TH. *The sum of the like odd powers of two quantities is exactly divisible by the sum of the quantities.*

Let x and a represent any two quantities, and let n be an *odd positive* integer; then $x^n + a^n$ is exactly divisible by $x + a$.

If n is odd, $\dfrac{x^n-(-a)^n}{x-(-a)}=\dfrac{x^n+a^n}{x+a}$ (77, Cor. 2).

But $x^n-(-a)^n$ is exactly divisible by $x-(-a)$; hence, x^n+a^n is exactly divisible by $x+a$.

The form of the quotient may be learned by finding a few of its terms by actual division, thus:

$$
\begin{array}{l|l}
x^n+a^n & \;x+a \\ \cline{2-2}
\underline{x^n+ax^{n-1}} & \;x^{n-1}-ax^{n-2}+a^2x^{n-3}-a^3x^{n-4}+\text{etc.} \\
\quad -ax^{n-1}+a^n & \\
\quad \underline{-ax^{n-1}-a^2x^{n-2}} & \\
\qquad\qquad +a^2x^{n-2}+a^n & \\
\qquad\qquad \underline{+a^2x^{n-2}+a^3x^{n-3}} & \\
\qquad\qquad\qquad\quad -a^3x^{n-3}+a^n & \\
\qquad\qquad\qquad\quad \underline{-a^3x^{n-3}-a^4x^{n-4}} & \\
\qquad\qquad\qquad\quad \;.\;\;.\;\;.\;\;.\;\;.\;\;. &
\end{array}
$$

$$\therefore\ \frac{x^n+a^n}{x+a}=x^{n-1}-ax^{n-2}+a^2x^{n-3}-a^3x^{n-4}+\ldots\ldots+a^{n-3}x^2-a^{n-2}x+a^{n-1},$$

when n is odd.

EXERCISES.

Write the quotients of

1. $\dfrac{x^3+a^3}{x+a}$.
2. $\dfrac{x^5+a^5}{x+a}$.
3. $\dfrac{x^7+a^7}{x+a}$.
4. $\dfrac{x^3+1}{x+1}$.
5. $\dfrac{x^5+1}{x+1}$.
6. $\dfrac{x^7+1}{x+1}$.
7. $\dfrac{a^9+b^9}{a+b}$.
8. $\dfrac{a^{11}+b^{11}}{a+b}$.
9. $\dfrac{x^{13}+y^{13}}{x+y}$.

CHAPTER III.

FACTORING.

92. Factoring is the process of resolving a quantity into its factors.

93. A **Composite Quantity** is a quantity which is exactly divisible by two or more integral quantities different from itself and unity. Thus, x^2-a^2 is a composite quantity, the factors of which are $x+a$ and $x-a$.

94. A **Prime Quantity** is a quantity which is exactly divisible only by itself and by 1. Thus, a, b, and $x+y$ are prime.

95. Two quantities are said to be *prime to each other*, or *relatively prime*, if they have no common factor. Thus, $2a$ and $3b$ are relatively prime.

When two quantities are said to have no common factor, it is understood that 1 is the only integral factor common to them.

96. To resolve a monomial into its prime factors.

Let it be required to resolve $12a^2b$ into its prime factors.

$$12a^2b = 2\cdot 2\cdot 3aab.$$

RULE.

To the prime factors of the numerical coefficient annex the prime factors of the literal part.

EXERCISES.

Resolve each of the following monomials into its prime factors:

1. $18a^2b^3$.
2. $27a^3b^2c$.
3. $21a^3b^3x^4$.
4. $49a^4b^3x^2y^3$.
5. $81m^2n^3x^5y^4$.
6. $-64p^5k^3$.
7. $-72a^2b^5c^3$.
8. $-210ax^3yz^2$.
9. $-(-26m^4x^2yz)$.

97. To resolve a polynomial into two factors, one of which shall be a monomial.

Let it be required to resolve $a + ax$ into two factors, one of which shall be a monomial.

$$\frac{a + ax}{a} = 1 + x;$$

$$\therefore \quad a + ax = a(1 + x) \quad \textbf{(16)}.$$

RULE.

Divide the given polynomial by any monomial that is a factor of each of its terms. The divisor will be one factor and the quotient the other.

EXERCISES.

Resolve each of the following expressions into two factors, one of which shall be a monomial:

1. $2a + 2ax$. *Ans.* $2a(1 + x)$, or $a(2 + 2x)$.
2. $axz + byz$. *Ans.* $z(ax + by)$.
3. $2x^2y + 3xy^2$. *Ans.* $xy(2x + 3y)$.
4. $6ab^2 + 12a^2bc$. *Ans.* $3ab(2b + 4ac)$.
5. $25a^4 - 40a^3b + 20a^2b^2$. *Ans.* $5a^2(5a^2 - 8ab + 4b^2)$.
6. $5a^2x + 10abx + 5b^2x$. *Ans.* $5x(a^2 + 2ab + b^2)$.
7. $48a^2b^2 - 8abcd + 16abd$. *Ans.* $8ab(6ab - cd + 2d)$.
8. $2ab^2c + 5ab^3 + 3ab^2c^2$. *Ans.* $ab^2(2c + 5b + 3c^2)$.
9. $3cx + 9cxz - 2cx^2$. *Ans.* $cx(3 + 9z - 2x)$.
10. $4c^4bx^3 - 5c^3x^4 + 2c^2x^3y$. *Ans.* $c^2x^3(4c^2b - 5cx + 2y)$.
11. $2abx^2 + 4abx + 2ab$. *Ans.* $2ab(x^2 + 2x + 1)$.

12. $3ab^2x^3 - 6ab^2x^2 + 12a^2b^3x$.

Ans. $3ab^2(x^3 - 2x^2 + 4abx)$.

13. $2x^3y + 6x^2y^2 + 6xy^3 + 2y^4$.

Ans. $2y(x^3 + 3x^2y + 3xy^2 + y^3)$.

14. $5ax^3 - 15ax^2y + 15axy^2 - 5ax^3$.

Ans. $5a(x^3 - 3x^2y + 3xy^2 - x^3)$.

15. $3cx^3 + 18cx^2 + 36cx + 24c$.

Ans. $3c(x^3 + 6x^2 + 12x + 8)$.

16. $x - 3x^2 + 6x^3 - 7x^4 + 6x^5 - 3x^6 + x^7$.

Ans. $x(1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6)$.

98. To resolve a binomial into its prime factors.

EXERCISES.

1. Resolve $x^2 - a^2$ into its prime factors.

$x^2 - a^2$ is exactly divisible by $x - a$ **(89)**.

$$\frac{x^2 - a^2}{x - a} = x + a;$$

whence, $$x^2 - a^2 = (x - a)(x + a).$$

2. Resolve $x^4 - a^4$ into its prime factors.

Two of the factors of $x^4 - a^4$ are $x - a$ and $x + a$ **(89, 90)**; hence the third factor may be found by dividing $x^4 - a^4$ by the product of $x - a$ and $x + a$.

$$\frac{x^4 - a^4}{(x - a)(x + a)} = \frac{x^4 - a^4}{x^2 - a^2} = x^2 + a^2;$$

whence, $$x^4 - a^4 = (x - a)(x + a)(x^2 + a^2).$$

3. Resolve $x^3 + a^3$ into its prime factors.

$x^3 + a^3$ is exactly divisible by $x + a$ **(91)**;

$$\frac{x^3 + a^3}{x + a} = x^2 - ax + a^2;$$

whence, $$x^3 + a^3 = (x + a)(x^2 - ax + a^2).$$

Resolve each of the following expressions into its prime factors:

4. $x^2 - 64$. *Ans.* $(x + 8)(x - 8)$.
5. $4x^2 - 81$. *Ans.* $(2x + 9)(2x - 9)$.
6. $36 - 25x^2$. *Ans.* $(6 + 5x)(6 - 5x)$.
7. $(a - b)^2 - c^2$. *Ans.* $(a - b + c)(a - b - c)$.
8. $x^5 + y^5$.
   Ans. $(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.
9. $a^8 - b^8$. *Ans.* $(a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$.
10. $1 - x^4$. *Ans.* $(1 + x^2)(1 + x)(1 - x)$.
11. $27x^3 + 1$. *Ans.* $(3x + 1)(9x^2 - 3x + 1)$.
12. $8x^3 - 1$. *Ans.* $(2x - 1)(4x^2 + 2x + 1)$.
13. $x^4 - a^6$. *Ans.* $(x^2 + a^3)(x^2 - a^3)$.
14. $(a + b)^4 - (c + d)^4$.
   Ans. $[(a + b)^2 + (c + d)^2](a+b+c+d)(a + b - c - d)$.

99. To resolve a trinomial of the form of

$$x^2 \pm 2ax + a^2$$

into its prime factors.

If two terms of a trinomial are positive squares, and the absolute value of the other term is twice the product of the square roots of these two, the trinomial is equal to the square of the SUM, *or the square of the* DIFFERENCE, *of these square roots, according as that other term is positive or negative.*

Thus,

$$x^2 \pm 2ax + a^2 = (x \pm a)^2 = (x \pm a)(x \pm a) \quad \textbf{(81, 82)}.$$

EXERCISES.

Resolve each of the following expressions into its prime factors:

1. $x^2 + 4x + 4$. *Ans.* $(x + 2)(x + 2)$.
2. $x^2 + 6x + 9$. *Ans.* $(x + 3)(x + 3)$.

3. $x^2 - 10x + 25$. *Ans.* $(x - 5)(x - 5)$.
4. $x^2 - 14x + 49$. *Ans.* $(x - 7)(x - 7)$.
5. $4x^2 + 36xy + 81y^2$. *Ans.* $(2x + 9y)(2x + 9y)$.
6. $49x^2 - 42xy + 9y^2$. *Ans.* $(7x - 3y)(7x - 3y)$.
7. $16a^6 - 8a^3 + 1$. *Ans.* $(4a^3 - 1)(4a^3 - 1)$.
8. $9x^4 - 6x^2y^2 + y^4$. *Ans.* $(3x^2 - y^2)(3x^2 - y^2)$.
9. $a^4 + b^4 + 2a^2b^2$. *Ans.* $(a^2 + b^2)(a^2 + b^2)$.
10. $x^6 + a^2 + 2ax^3$. *Ans.* $(x^3 + a)(x^3 + a)$.
11. $(a + b)^2 - 2(a + b)(c + d) + (c + d)^2$.
Ans. $(a + b - c - d)(a + b - c - d)$.

100. To resolve a trinomial of the form of

$$x^2 \pm mx \pm n$$

into its prime factors.

The following formulas may be verified by performing the indicated multiplication:

$$x^2 + (a + b)x + ab = (x + a)(x + b),$$
$$x^2 - (a + b)x + ab = (x - a)(x - b),$$
$$x^2 + (a - b)x - ab = (x + a)(x - b),$$
$$x^2 - (a - b)x - ab = (x - a)(x + b).$$

It will be observed that, in each of these equations, the coefficient of x in the second term of the first member is equal to the sum of two quantities whose product is equal to the third term. Thus, in the second equation, the coefficient of x in the first member is $-a - b$, and the product of $-a$ and $-b$ is equal to the third term. Hence,

Any trinomial of the form of $x^2 \pm mx \pm n$ *can be resolved into two binomial factors, if the coefficient in the second term is equal to the sum of two quantities whose product is equal to the third term.*

EXERCISES.

1. Resolve $x^2 + 5x + 6$ into its prime factors.

If two numbers whose sum is 5 and product 6 can be found, this expression can be factored. It is evident that 2 and 3 are the numbers;

$$\therefore \quad x^2 + 5x + 6 = (x + 2)(x + 3).$$

2. Resolve $x^2 - 12x + 32$ into its prime factors.

$$-8 - 4 = -12, \quad \text{and} \quad -8 \times (-4) = 32;$$

$$\therefore \quad x^2 - 12x + 32 = (x - 8)(x - 4).$$

3. Resolve $x^2 + 2x - 35$ into its prime factors.

$$-5 + 7 = 2, \quad \text{and} \quad -5 \times 7 = -35;$$

$$\therefore \quad x^2 + 2x - 35 = (x - 5)(x + 7).$$

4. Resolve $x^2 - 3x - 54$ into its prime factors.

$$-9 + 6 = -3, \quad \text{and} \quad -9 \times 6 = -54;$$

$$\therefore \quad x^2 - 3x - 54 = (x - 9)(x + 6).$$

Resolve each of the following expressions into its prime factors:

5.	$x^2 + 8x + 15.$	*Ans.* $(x + 5)(x + 3).$
6.	$x^2 + 10x + 21.$	*Ans.* $(x + 7)(x + 3).$
7.	$x^2 - 2x - 24.$	*Ans.* $(x + 4)(x - 6).$
8.	$a^2 - 20a - 21.$	*Ans.* $(a + 1)(a - 21).$
9.	$a^2 + 14a - 51.$	*Ans.* $(a + 17)(a - 3).$
10.	$a^2 - a - 42.$	*Ans.* $(a + 6)(a - 7).$
11.	$a^2 - 12ab - 13b^2.$	*Ans.* $(a + b)(a - 13b).$
12.	$a^2 + 10ab - 11b^2.$	*Ans.* $(a - b)(a + 11b).$
13.	$a^2 - a - 2.$	*Ans.* $(a + 1)(a - 2).$
14.	$a^2 - a - 30.$	*Ans.* $(a + 5)(a - 6).$

101. To resolve a trinomial of the form of

$$x^{2c} \pm mx^c \pm n$$

into its prime factors.

Since $x^{2c} + (a + b)\, x^c + ab = (x^c + a)\, (x^c + b)$,

$x^{2c} - (a + b)\, x^c + ab = (x^c - a)\, (x^c - b)$,

$x^{2c} + (a - b)\, x^c - ab = (x^c + a)\, (x^c - b)$,

and $x^{2c} - (a - b)\, x^c - ab = (x^c - a)\, (x^c + b)$,

such expressions as

$$x^4 + 8x^2 + 15, \qquad x^8 + 4x^4 - 32,$$
$$x^6 - 9x^3 + 20, \qquad x^{12} - 5x^6 - 66,$$

may be resolved into prime factors by a method similar to that of the preceding article. Thus,

1. $x^4 + 8x^2 + 15 = (x^2 + 3)\, (x^2 + 5)$.
2. $x^6 - 9x^3 + 20 = (x^3 - 4)\, (x^3 - 5)$.
3. $x^8 + 6x^4 - 16 = (x^4 + 8)\, (x^4 - 2)$.
4. $x^{12} - 5x^6 - 66 = (x^6 + 6)\, (x^6 - 11)$.
5. Resolve $x^{10} - 12x^5 + 27$ into its prime factors. *Ans.* $(x^5 - 9)\, (x^5 - 3)$.
6. Resolve $x^{10} + 8x^5 - 20$ into its prime factors. *Ans.* $(x^5 + 10)\, (x^5 - 2)$.

MISCELLANEOUS EXERCISES.

102. 1. Resolve $4b^2c^2 - (b^2 + c^2 - a^2)^2$ into its prime factors.

$4b^2c^2 - (b^2 + c^2 - a^2)^2 = (2bc + b^2 + c^2 - a^2)\, (2bc - b^2 - c^2 + a^2)$ **(83).**

But $2bc + b^2 + c^2 - a^2 = (b + c)^2 - a^2$

$= (b + c + a)\, (b + c - a)$,

and $2bc - b^2 - c^2 + a^2 = a^2 - (b^2 - 2bc + c^2)$

$= a^2 - (b - c)^2$

$= (a + b - c)\, (a - b + c)$.

$\therefore\ 4b^2c^2 - (b^2 + c^2 - a^2)^2$

$= (b + c + a)\, (b + c - a)\, (a + b - c)\, (a - b + c)$.

2. Resolve $x^2 + 2xy + y^2 - a^2 + 2ab - b^2$ into its prime factors.

$$x^2 + 2xy + y^2 = (x + y)^2,$$

and $$-a^2 + 2ab - b^2 = -(a - b)^2;$$

$$\begin{aligned} \therefore\quad x^2+2xy+y^2-a^2+2ab-b^2 &= (x + y)^2 - (a - b)^2 \\ &= (x + y + a - b)(x + y - a + b). \end{aligned}$$

3. Resolve $ac + ad + bd + bc$ into its prime factors.

$$\begin{aligned} ac + ad + bd + bc &= a(c + d) + b(c + d) \\ &= (a + b)(c + d). \end{aligned}$$

4. Resolve $7x^2 - 12x + 5$ into its prime factors.

$$\begin{aligned} 7x^2 - 12x + 5 &= 7x^2 - 5x - 7x + 5 \\ &= 7x^2 - 5x - (7x - 5) \\ &= x(7x - 5) - (7x - 5) \\ &= (x - 1)(7x - 5). \end{aligned}$$

5. Resolve $x^4 - 17x^2 + 16$ into its prime factors.

$$\begin{aligned} x^4 - 17x^2 + 16 &= (x^2 - 16)(x^2 - 1) \\ &= (x+4)(x-4)(x+1)(x-1). \end{aligned}$$

Resolve each of the following expressions into its prime factors:

6. $a^3 - a.$ *Ans.* $(a - 1)(a + 1)a.$
7. $5ax^2 + 10axy + 5ay^2.$ *Ans.* $5a(x + y)(x + y).$
8. $3ax^2 - 18ax + 27a.$ *Ans.* $3a(x - 3)(x - 3).$
9. $36a - 24ax + 4ax^2.$ *Ans.* $2 \cdot 2a(3 - x)(3 - x).$
10. $3a^3b - 3ab^3.$ *Ans.* $3ab(a + b)(a - b).$
11. $3x^2 + 9x - 12.$ *Ans.* $3(x + 4)(x - 1).$
12. $5x^3 + 10x^2 - 175x.$ *Ans.* $5x(x + 7)(x - 5).$
13. $ac + ad - bd - bc.$ *Ans.* $(a - b)(c + d).$
14. $ab + 2bx + 2ax + b^2.$ *Ans.* $(a + b)(b + 2x).$

15. $x^3 - xy^2$. *Ans.* $x(x + y)(x - y)$.
16. $5x^2 - 7x + 2$. *Ans.* $(x - 1)(5x - 2)$.
17. $x^3 - x^2 - 2x$. *Ans.* $x(x + 1)(x - 2)$.
18. $9 + x^4 - 10x^2$. *Ans.* $(x+3)(x-3)(x+1)(x-1)$.
19. $16 - 17x^2 + x^4$. *Ans.* $(x+4)(x-4)(x+1)(x-1)$.
20. $(a - b)(b - c) + (a - d)(c - d)$.
Ans. $(a - b + c - d)(b - d)$.
21. $x^4 - xy^3$. *Ans.* $x(x - y)(x^2 + xy + y^2)$.
22. $a^8 - b^4$. *Ans.* $(a^4 + b^2)(a^2 + b)(a^2 - b)$.
23. $xy^4 + x^4y$. *Ans.* $xy(x + y)(x^2 - xy + y^2)$.
24. $2x^3y^2z - 8xy^2z^3$. *Ans.* $2xy^2z(x + 2z)(x - 2z)$.
25. $a^4 - 4b^4$. *Ans.* $(a^2 - 2b^2)(a^2 + 2b^2)$.
26. $x^4 + x^2y^2 + y^4$. *Ans.* $(x^2+xy+y^2)(x^2-xy+y^2)$.
27. $a^4 - 2a^2b^2 + b^4$. *Ans.* $(a + b)^2(a - b)^2$.
28. $a^2 + b^2 - c^2 + 2ab$. *Ans.* $(a+b+c)(a+b-c)$.
29. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd$.
Ans. $(a + b + c + d)(a + b - c - d)$.
30. $a^2 - b^2 - c^2 + d^2 + 2bc + 2ad$.
Ans. $(a + b - c + d)(a - b + c + d)$.
31. $a^2 - (b - c)^2$. *Ans.* $(a + b - c)(a - b + c)$.
32. $(x + 7)^2 - (x + 2)^2$. *Ans.* $5(2x + 9)$.
33. $x^2 - 3x - 70$. *Ans.* $(x - 10)(x + 7)$.
34. $x^2 + 11x + 10$. *Ans.* $(x + 1)(x + 10)$.
35. $a^2 - 15ab + 56b^2$. *Ans.* $(a - 7b)(a - 8b)$.
36. $x^2 - 4x - 192$. *Ans.* $(x - 16)(x + 12)$.
37. $a^2x^2 + abxy - 42b^2y^2$.
Ans. $(ax + 7by)(ax - 6by)$.
38. $3ax^2 - 24ax - 60a$. *Ans.* $3a(x - 10)(x + 2)$.
39. $-24ac - 5ac^2 + ac^3$. *Ans.* $ac(c - 8)(c + 3)$.
40. $6x^2 - 11x - 35$. *Ans.* $(3x + 5)(2x - 7)$.

41. $8x^2 + 6x - 135$. *Ans.* $(2x + 9)(4x - 15)$.

42. $18x^2 - 21x - 72$. *Ans.* $3(2x + 3)(3x - 8)$.

43. $20x^2 - 11x - 42$. *Ans.* $(4x - 7)(5x + 6)$.

44. $3x^3y + 10x^2y^2 + 3xy^3$. *Ans.* $xy(3x+y)(x+3y)$.

45. $20x^3 + 12ax^2 + 25bx^2 + 15abx$.
Ans. $x(5x + 3a)(4x + 5b)$.

46. $m^2x^2 + (mq + mp)x + pq$.
Ans. $(mx + p)(mx + q)$.

47. $a^2(b - c) + b^2(c - a) + c^2(a - b)$.
Ans. $(a - b)(b - c)(a - c)$.

48. $x^6 - y^6$.
Ans. $(x^2+xy+y^2)(x^2-xy+y^2)(x+y)(x-y)$.

49. $x^2y - y^9$. *Ans.* $(x + y^4)(x - y^4)y$.

50. $x^2 - 2xy - 3y^2 + 4yz - z^2$.
Ans. $(x - 3y + z)(x + y - z)$.

51. $(a + b)^2 - 4b^2$. *Ans.* $(a + 3b)(a - b)$.

52. $(2a + b)^2 - (2a - b + c)^2$.
Ans. $(4a + c)(2b - c)$.

53. $m^3 - n^3 - m(m^2 - n^2) + n(m - n)^2$.
Ans. $mn(m - n)$.

54. $2(a^3 + a^2b + ab^2) - (a^3 - b^3)$.
Ans. $(a + b)(a^2 + ab + b^2)$.

55. $2x^3y + 5x^2y^2 + 2xy^3$. *Ans.* $xy(2x+y)(x+2y)$.

56. $6b^2x^2 - 7bx^3 - 3x^4$. *Ans.* $x^2(2b - 3x)(3b + x)$.

CHAPTER IV.

GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE.

GREATEST COMMON DIVISOR.

103. A **Common Divisor** of two or more quantities is a quantity that will exactly divide them. Thus, a, b, and ab are common divisors of a^2b^3 and abc.

Any integral factor common to two or more quantities is a common divisor of them.

A common divisor is sometimes called a *Common Measure.*

104. The **Greatest Common Divisor** of two or more quantities is that common divisor of them which contains the *greatest* number of prime factors. Thus, $3a^2x$ is the greatest common divisor of $9a^2bx^2$ and $12a^4c^2xz$.

105. To find the g. c. d. of two or more expressions by factoring.

1. Let it be required to find the g. c. d. of $4a^3b^3x$, $6a^4b^2c^2$, and $10a^5b^4d^3$.

$$4a^3b^3x = 2^2a^3b^3x,$$

$$6a^4b^2c^2 = 3\cdot 2a^4b^2c^2,$$

$$10a^5b^4d^3 = 5\cdot 2a^5b^4d^3.$$

The g. c. d. of two or more expressions must contain all the prime factors that are common, and no others; hence the g. c. d. of the given expressions is $2a^3b^2$.

2. Let it be required to find the g. c. d. of $4am^2 + 4bm^2$ and $6amn + 6bmn$.

$$4am^2 + 4bm^2 = 2^2m^2(a + b),$$

and $$6amn + 6bmn = 3 \cdot 2mn(a + b);$$

hence, $2m(a + b)$ is the g. c. d.

RULE.

Find the product of all the prime factors common to the given expressions.

Cor. The exponent of any factor in the g. c. d. of two or more expressions is equal to the exponent of the same factor in that expression in which its exponent is either less or not greater than in any other.

EXERCISES.

Find the g. c. d. of

1. $6a^2b$ and $9ab^2$. *Ans.* $3ab$.
2. $10a^3bc$ and $18ab^3c$. *Ans.* $2abc$.
3. $32a^2b^3c^2$ and $48ab^4c$. *Ans.* $16ab^3c$.
4. $51a^2xyz^2$ and $111ax^3y^3z$. *Ans.* $3axyz$.
5. $3ab$, $5a^2b$, $15ab^3$. *Ans.* ab.
6. $21a^3x$, $49ax^3y$, $9xy^3$. *Ans.* x.
7. $2x - 2a$ and $3x^2 - 3a^2$. *Ans.* $x - a$.
8. $2(x + a)$, $8(x^2 - 2ax + x^2)$, $4(x^2 - a^2)$. *Ans.* 2.
9. $x^2 - b^2$ and $3(x^2 + 2bx + b^2)$. *Ans.* $x + b$.
10. $x^3 - a^3$ and $x^2 - a^2$. *Ans.* $x - a$.
11. $x^2 + ax + a^2$ and $3x^3 - 3a^3$. *Ans.* $x^2 + ax + a^2$.
12. $x^2 + 9ax + 14a^2$ and $x^2 - 4a^2$. *Ans.* $x + 2a$.
13. $4x^3(a + x)^2$ and $10(a^2x - x^3)^2$. *Ans.* $2x^2(a + x)^2$.

14. $x^2(a^2 - x^2)^2$ and $(a^2x + ax^2)^5$. *Ans.* $x^2(a + x)^2$.

15. $(a^2b - ab^2)^2$ and $ab(a^2 - b^2)^2$. *Ans.* $ab(a - b)^2$.

16. $6(x^2 - 1)$ and $8(x^2 - 3x + 2)$. *Ans.* $2(x - 1)$.

17. $(x^2 + x)^2$ and $x^3(x^2 - x - 2)$. *Ans.* $x^2(x + 1)$.

18. $4(x^3 + a^3)$ and $6(x^2 - 2ax - 3a^2)$.
Ans. $2(x + a)$.

19. $a^3(x^2 + 12x + 11)$ and $a^2x^2 - 11a^2x - 12a^2$.
Ans. $a^2(x + 1)$.

20. $9(a^2x^2 - 4)$ and $12(a^2x^2 + 4ax + 4)$.
Ans. $3(ax + 2)$.

21. $a^3 - b^3$ and $a^3 + a^2b + ab^2$. *Ans.* $a^2 + ab + b^2$.

22. $x^2 + (a + b)x + ab$ and $x^2 + (a + c)x + ac$.
Ans. $x + a$.

106. To find the g. c. d. of two expressions by division.

Let A and B represent two expressions whose g. c. d. is to be found.

I. If P is a divisor of A, it is a divisor of mA, if m is an integer.

Suppose $\frac{A}{P} = a$; then $A = aP$ (**64**, 3), and $mA = maP$.

Now P is a divisor of maP; hence it is a divisor of mA, for $mA = maP$.

II. If P is a divisor of A and B, it is a divisor of $mA \pm nB$, if m and n are integers.

Suppose $\frac{A}{P} = a$, and $\frac{B}{P} = b$; then $A = aP$, and $B = bP$;

$$\therefore \quad mA \pm nB = maP \pm nbP = (ma \pm nb)P.$$

Therefore P is a divisor of $mA \pm nB$.

III. Suppose A and B to be arranged according to the powers of some common letter, and the greatest exponent of that letter in A to be not less than the greatest exponent of the same letter in B. Divide A by B; let p denote the quotient, and C, the remainder: divide B by C; let q denote the quotient, and D, the remainder: divide C by D; let r denote the quotient, and suppose there is no remainder. The last divisor, D, is the g c. d. required. This may be proved as follows:

$$\begin{array}{r|l} A & \underline{B} \\ \underline{pB} & p \end{array} \qquad \begin{array}{r|l} B & \underline{C} \\ \underline{qC} & q \end{array} \qquad \begin{array}{r|l} C & \underline{D} \\ \underline{rD} & r. \end{array}$$

If the remainder be added to the product of divisor and quotient, the sum will be equal to the dividend;

$$\therefore \quad A = pB + C,$$

and
$$B = qC + D.$$

Now, because D is a divisor of C, it is a divisor of qC (I); therefore, D is a divisor of $qC+D$ (II); that is, D is a divisor of B. Again, since D is a divisor of B and C, it is a divisor of $pB+C$; that is, D is a divisor of A. Hence, D is a *common* divisor of A and B.

If the product of divisor and quotient be subtracted from the dividend, the result will be the remainder;

$$\therefore \quad A - pB = C, \quad \text{and} \quad B - qC = D.$$

Now, *every* common divisor of A and B is a divisor of $A - pB$, or C (II); hence, every common divisor of A and B is a common divisor of B and C. Again, every common divisor of B and C is a divisor of $B - qC$, or D. It has thus been proved (1) that D is a common divisor of A and B, and (2) that every common divisor of A and B is a divisor of D. Then, since no expression of a higher degree than D is a divisor of D, it follows that D is the g. c. d. required.

Illustration. Let it be required to find the g. c. d. of $4x^3 - 21x^2 + 15x + 20$ and $x^2 - 6x + 8$.

$$
\begin{array}{r|l}
4x^3 - 21x^2 + 15x + 20 & x^2 - 6x + 8 \\ \cline{2-2}
4x^3 - 24x^2 + 32x & 4x + 3 \\ \cline{1-1}
3x^2 - 17x + 20 & \\
3x^2 - 18x + 24 & \\ \cline{1-1}
\end{array}
$$

$$
\begin{array}{r|l}
x^2 - 6x + 8 & x - 4 \\ \cline{2-2}
x^2 - 4x & x - 2 \\ \cline{1-1}
-2x + 8 & \\
-2x + 8. & \\ \cline{1-1}
\end{array}
$$

Dividing $4x^3 - 21x^2 + 15x + 20$ by $x^2 - 6x + 8$, the quotient is found to be $4x + 3$, and the remainder $x - 4$. Dividing $x^2 - 6x + 8$ by $x - 4$, the quotient is found to be $x - 2$, and the remainder 0. Hence, $x - 4$ is the g. c. d.

IV. Suppose one of the given expressions, whose g. c. d. is to be found, and one factor of the other to be relatively prime; then that factor may be rejected. Thus, suppose $B = mS$, and that m has no factor that A has; then m may be rejected; that is, the g. c. d. of A and B is the same as that of A and S. For no factor *common* to A and B is removed by the rejection of m, since m contains no factor common to them. In like manner it may be shown that, if at any stage of the process a dividend or divisor contains a factor which has no factor common to them, that factor may be rejected.

Illustration. Let it be required to find the g. c. d. of $x^4 + 7x^3 - 9x^2 - 7x + 8$ and $6x^4 - 54x^3 + 42x^2 + 54x - 48$.

$$6x^4 - 54x^3 + 42x^2 + 54x - 48 = (x^4 - 9x^3 + 7x^2 + 9x - 8)6.$$

$$
\begin{array}{r|l}
x^4 + 7x^3 - 9x^2 - 7x + 8 & x^4 - 9x^3 + 7x^2 + 9x - 8 \\ \cline{2-2}
x^4 - 9x^3 + 7x^2 + 9x - 8 & 1 \\ \cline{1-1}
16x^3 - 16x^2 - 16x + 16 & = (x^3 - x^2 - x + 1)16.
\end{array}
$$

$$\begin{array}{r|l}
x^4 - 9x^3 + 7x^2 + 9x - 8 & x^3 - x^2 - x + 1 \\ \cline{2-2}
x^4 - x^3 - x^2 + x \phantom{{}-8} & x - 8 \\ \cline{1-1}
- 8x^3 + 8x^2 + 8x - 8 & \\
- 8x^3 + 8x^2 + 8x - 8. & \\ \cline{1-1}
\end{array}$$

Rejecting the factor 6 from the second expression, and dividing the first by the result, the quotient is found to be 1, and the remainder, $16x^3 - 16x^2 - 16x + 16$. Rejecting the factor 16 from the remainder, and dividing the first divisor by the result, the quotient is found to be $x - 8$, and the remainder, 0. Hence, $x^3 - x^2 - x + 1$ is the g. c. d.

V. Suppose m has no factor that B has, and n no factor that A has; then the g. c. d. of A and B is the same as that of mA and B or of A and nB. For A and B have the same common factors that mA and B have, and the same that A and nB have, since m has no factor that B has, and n no factor that A has. In like manner it may be shown that, at any stage of the process, a factor may be introduced into one of the expressions, used as dividend and divisor, provided that factor and the other expression are relatively prime.

ILLUSTRATION. Let it be required to find the g. c. d. of $x^3 - x^2 - 2x + 2$ and $2x^2 - 5x + 3$.

$$\begin{array}{r|l}
2x^3 - 2x^2 - 4x + 4 & 2x^2 - 5x + 3 \\ \cline{2-2}
2x^3 - 5x^2 + 3x \phantom{{}+4} & x, \quad + 3 \\ \cline{1-1}
3x^2 - 7x + 4. & \\
& \\
6x^2 - 14x + 8 & \\
6x^2 - 15x + 9 & \\ \cline{1-1}
\end{array}$$

$$\begin{array}{r|l}
2x^2 - 5x + 3 & x - 1 \\ \cline{2-2}
2x^2 - 2x \phantom{{}+3} & 2x - 3 \\ \cline{1-1}
- 3x + 3 & \\
- 3x + 3. & \\ \cline{1-1}
\end{array}$$

Multiplying the first expression by 2, and dividing the product by the second, the quotient is found to be x and the remainder, $3x^2 - 7x + 4$. Multiplying this remainder by 2, and dividing the product by the divisor, the quotient is found to be 3, and the remainder, $x - 1$. Dividing $2x^2 - 5x + 3$ by $x - 1$, the quotient is found to be $2x - 3$, and the remainder, 0. Hence, $x - 1$ is the g. c. d.

RULE.

1. *Arrange the given expression according to the powers of a common letter, and reject any monomial factor in either that is prime to the other.*

2. *Apply the process of division to the results, using that expression which is not of a lower degree than the other as a dividend and the other as a divisor; continue the division until a remainder of a lower degree than the divisor is found; then divide the divisor by this remainder in the same way: again, divide the second divisor by the second remainder: proceed in this way until there is no remainder; then the last divisor will be the g. c. d. required: provided the first term of each dividend is exactly divisible by the first term of each divisor.*

3. *But, if the first term of any dividend is not exactly divisible by the first term of the corresponding divisor, avoid fractional terms in the quotient by rejecting or introducing suitable factors* (IV, V), *or by both operations, if necessary.*

Cor. 1. If at any stage of the process of finding the g. c. d. of two expressions, a factor is seen to be common to a dividend and its corresponding divisor, that factor may be set aside as one factor of the g.c.d. Thus, suppose $A = aF$, and $B = bF$; then the g. c. d. of A and B may be found by finding the g. c. d. of a and b, and multiplying the result by F.

COR. 2. Every common divisor of A and B is a divisor of their g. c. d.; and every divisor of their g. c. d. is a common divisor of A and B.

EXERCISES.

Find the g. c. d. of

1. $2x^2 - x - 10$ and $3x^2 + 13x + 14$. *Ans.* $x + 2$.
2. $21x^2 + 8x - 45$ and $12x^2 + 5x - 25$. *Ans.* $3x + 5$.
3. $12a^2 - 2ab - 70b^2$ and $8a^2 - 2ab - 45b^2$. *Ans.* $2a - 5b$.
4. $3x^2 - 9xy - 30y^2$ and $x^2 - 3xy - 10y^2$. *Ans.* $x^2 - 3xy - 10y^2$.
5. $18x^2 + 12xy - 70y^2$ and $21x^2 - 41xy + 10y^2$. *Ans.* $3x - 5y$.
6. $18x^3 - 63x^2 - 30x + 105$ and $9x^3 + 15x^2 - 15x - 25$. *Ans.* $3x^2 - 5$.
7. $x^2 + 12x + 35$ and $x^2 + 13x + 42$. *Ans.* $x + 7$.
8. $x^3 + 3x^2 + 4x + 4$ and $x^3 + 4x^2 + 5x + 6$. *Ans.* $x^2 + x + 2$.
9. $2x^2 + 18x + 36$ and $2x^3 + 20x^2 + 48x$. *Ans.* $2(x + 6)$.
10. $36x^4 + 30x^3 - 5x - 1$ and $6x^3 + 17x^2 + 11x + 2$. *Ans.* $6x^2 + 5x + 1$.
11. $6x^3 - 5x^2 - 13x + 12$ and $12x^3 - 8x^2 - 35x + 6$. *Ans.* $2x + 3$.
12. $2x^3 + 3x^2 - 11x + 3$ and $3x^4 + 9x^3 - 7x^2 - 12x + 4$. *Ans.* $x^2 + 3x - 1$.
13. $3x^5 + 2x^4 + x^2$ and $3x^4 + 2x^3 - 3x^2 + 2x - 1$. *Ans.* $3x^2 - x + 1$.
14. $a^3 - b^3 + c^3 + 3abc$ and $a^2 - b^2 + c^2 + 2ac$. *Ans.* $a - b + c$.

107. To find the g.c.d. of three or more expressions by division.

Let A, B, C, D, etc., represent the given expressions.

Find the g. c. d. of two of these, as A and B. Let d represent this g. c. d.; then the g. c. d. of C and d will be the g. c. d. of A, B, and C. For every common divisor of C and d is a common divisor of A, B, and C (**106**, Cor. 2); and every common divisor of A, B, and C is a common divisor of C and d. Hence, the g. c. d. of C and d is the g. c. d. of A, B, and C.

Let e represent the g. c. d. of A, B, and C; then it may be shown, in the same manner, that the g. c. d. of e and D is the g. c. d. of A, B, C, and D; and so on.

RULE.

Find the g. c. d. of any two of the given expressions; then find the g. c. d. of the result and a third expression; and so on. The final result will be the g. c. d. required.

EXERCISES.

Find the g. c. d. of

1. $x^3 + 5x^2 + 2x - 12$, $2x^3 - x^2 - 18x + 20$, $x^4 + 8x^3 + 17x^2 - 6x - 36$. *Ans.* $x^2 + 2x - 4$.

2. $2x^2 + x - 3$, $3x^2 - 2x - 1$, $x^3 - x^2 - x + 1$. *Ans.* $x - 1$.

3. $4x^2 - 2x - 6$, $2x^2 - 5x + 3$, $6x^2 - 13x + 6$. *Ans.* $2x - 3$.

4. $9x^2 - 9x - 10$, $6x^2 + 2x - 20$, $3x^2 - 26x + 35$. *Ans.* $3x - 5$.

5. $x^2 - 9$, $x^2 - 6x + 9$, $3x^3 + 9x^2 - 27x - 81$. *Ans.* $x - 3$.

6. $x^2 - 3xy - 10y^2$, $x^2 + 2xy - 35y^2$, $3x^2 - 24xy + 45y^2$. *Ans.* $x - 5y$.

7. $2x^2 + 5x - 7$, $3x^2 - 2x - 1$, and $x^3 - x^2 - x + 1$.
Ans. $x - 1$.

8. $2x^2 - x - 3$, $2x^2 - 5x + 3$, and $6x^2 - 13x + 6$.
Ans. $2x - 3$.

9. $9x^2 - 18x + 5$, $3x^2 + x - 10$, and $3x^2 - 26x + 35$.
Ans. $3x - 5$.

LEAST COMMON MULTIPLE.

108. If one expression is exactly divisible by another, the first is called a **Multiple** of the other. Thus, 12 is a multiple of 3, and $5a^2b$ is a multiple of a.

109. A **Common Multiple** of two or more expressions is an expression which is exactly divisible by each of them. Thus, 27 is a common multiple of 3 and 9, and $10ab$ is a common multiple of $5a$ and $2b$.

110. The **Least Common Multiple** of two or more expressions is that common multiple of them which contains the *least* number of prime factors. Thus, 12 is the least common multiple of 4 and 3, and $10a^2b^2$ is the least common multiple of $5a^2$ and $2b^2$.

111. To find the l. c. m. of two or more expressions by factoring.

Let it be required to find the l. c. m. of $4a^2bx^3$, $21abx^3y$, and $12ab^2xy^3$.

$$4a^2bx^3 = 2^2a^2bx^3,$$

$$21abx^3y = 7 \cdot 3abx^3y,$$

and $$12ab^2xy^3 = 2^2 \cdot 3ab^2xy^3.$$

Multiplying $4a^2bx^3$ by $21y$, which is the product of the prime factors of $21abx^3y$ that are not found in $4a^2bx^3$, the result is $84a^2bx^3y$. This is the l. c. m. of $4a^2bx^3$ and $21abx^3y$; for it contains all the prime factors of each of them, and no other prime factors. Again, multiplying $84a^2bx^3y$ by by^2, which is the product of the prime factors of $12ab^2xy^3$ that are not found in $84a^2bx^3y$, the result is $84a^2b^2x^3y^3$. This is the l. c. m. of the three given expressions; for it contains all the prime factors of each of them, and no other prime factors.

RULE.

Find the product of one of the given expressions and all the prime factors of another that are not already in that expression; then find the product of the result and all the prime factors of a third that are not already in that result; and so on. The final result will be the l. c. m. required.

COR. 1. If no two of the given expressions have a common factor, their l. c. m. is equal to their product.

COR. 2. The exponent of any factor in the l. c. m. of two or more expressions is equal to the exponent of the same factor in that expression in which its exponent is either greater or not less than in any other.

COR. 3. Every multiple of the l. c m. of two or more expressions is a common multiple of them.

COR. 4. Every common multiple of two or more expressions is a multiple of their l. c. m.

EXERCISES.

Find the l. c. m. of

1. $6a^2b$ and $9ab^2$. *Ans.* $18a^2b^2$.
2. $10x^3yz$ and $18xy^3z$. *Ans.* $90x^3y^3z$.
3. $3ab$, $5a^2b$, and $15ab^3$. *Ans.* $15a^2b^3$.
4. $2(x-a)$ and $3(x^2-a^2)$. *Ans.* $6(x^2-a^2)$.

5. $2(x+a)$, $8(x-a)^2$, $4(x^2-a^2)$.
Ans. $8(x-a)^2(x+a)$.

6. $a(b^2-c^2)$, $d(b+c)$, $e(b-c)$.
Ans. $ade(b^2-c^2)$.

7. x^2+ax+a^2, $3(x^3-a^3)$, $x-a$.
Ans. $3(x^3-a^3)$.

8. x^2+ax+a^2, $5(x-a)$, $6(x^3+a^3)$.
Ans. $30(x^6-a^6)$.

9. $21(x^4-a^4)$, $14(x^2+a^2)$, $18(x^2-a^2)$.
Ans. $126(x^4-a^4)$.

10. $x^2-6ax+9a^2$, $x^2+ax-12a^2$, $x+4a$.
Ans. $(x-3a)^2(x+4a)$.

11. $x^2+9ax+14a^2$ and x^2-4a^2.
Ans. $(x^2-4a^2)(x+7a)$.

12. a^2-ax+x^2, a^2+ax+x^2, a^3+x^3, a^3-x^3.
Ans. a^6-x^6.

13. $a(x+1)$, $b(x^2-1)$, $c(x^2+2x-3)$, $d(x^2+4x+3)$.
Ans. $abcd(x^2-1)(x+3)$.

112. To find the l. c. m. of two expressions by means of their g. c. d.

Let it be required to find the l.c.m. of $4a^2bx^3$ and $24abx^3y$. The g. c. d. of these expressions is $4abx^3$. Dividing $24abx^3y$ by $4abx^3$, the quotient is found to be $6y$. Now, since $4abx^3$ contains all the prime factors common to the given expressions, it follows that $6y$ contains all the prime factors of $24abx^3y$ that are not found in $4a^2bx^3$. Multiplying $4a^2bx^3$ by $6y$, the product is found to be $24a^2bx^3y$, which is the l. c. m. required **(111)**.

RULES.

I. *Divide one of the given expressions by their g. c. d., and multiply the quotient thus obtained by the other.* Or,

II. *Divide the product of the two expressions by their g. c. d.*

EXERCISES.

Find the l. c. m. of

1. $2x^2 - x - 10$ and $3x^2 + 13x + 14$.
   Ans. $(2x - 5)(3x^2 + 13x + 14)$.
2. $5(6x^2 - x - 1)$ and $2x^2 + 3x - 2$.
   Ans. $5(3x + 1)(2x^2 + 3x - 2)$.
3. $3(x^3 - 1)$ and $5(x^2 + x - 2)$.
   Ans. $15(x + 2)(x^3 - 1)$.
4. $2x^3 - 18x^2 + 46x - 30$ and $3x^2 - 24x + 21$.
   Ans. $6(x^3 - 9x^2 + 23x - 15)(x - 7)$.
5. $6x^2 - 10x + 4$ and $12x^3 - 12x^2 - 3x + 3$.
   Ans. $6(4x^3 - 4x^2 - x + 1)(3x - 2)$.
6. $x^3 + 2ax^2 - a^2x - 2a^3$ and $x^3 - 2ax^2 - a^2x + 2a^3$.
   Ans. $(x^2 - a^2)(x^2 - 4a^2)$.

113. To find the l. c. m. of three or more expressions.

Let A, B, C, D, etc., represent the given expressions. Find the l. c. m. of two of these, as A and B. Let m represent this l. c. m.; then the l. c. m. of m and C will be the l. c. m. of A, B, and C. For every common multiple of m and C is a common multiple of A, B, and C (**111**, Cor. 3); and every common multiple of A, B, and C is a multiple of m and C (**111**, Cor. 4). Hence, the l. c. m. of m and C is the l. c. m. of A, B, and C.

Let n represent the l. c. m. of A, B, and C; then it may be shown, in the same manner, that the l. c. m. of n and D is the l. c. m. of A, B, C, and D; and so on.

RULE.

Find the l. c. m. of any two of the given expressions; then find the l. c. m. of the result and a third expression; and so on. The final result will be the l. c. m. required.

EXERCISES.

Find the l. c. m. of

1. $2a - 1,\ 4a^2 - 1,\ 4a^2 + 1.$ *Ans.* $16a^4 - 1.$

2. $x^3 - 1,\ 2x^3 - 2x,\ 3x^3 + 3.$ *Ans.* $6x(x^6 - 1).$

3. $x^2 + 7x + 10,\ x^2 - 2x - 8,\ 2x^2 + 2x - 40.$
Ans. $2x^3 + 6x^2 - 36x - 80.$

4. $x^2 - 3xy + 2y^2,\ x^2 - xy - 2y^2,\ x^2 - y^2.$
Ans. $x^3 - 2x^2y - xy^2 + 2y^3.$

5. $2x^2 + x - 3,\ 3x^2 - 2x - 1,\ x^3 - x^2 - x + 1.$
Ans. $(x - 1)^2(x + 1)(2x + 3)(3x + 1).$

6. $2x^2 - x - 3,\ 2x^2 - 5x + 3,\ 6x^2 - 13x + 6.$
Ans. $(x - 1)(x + 1)(2x - 3)(3x - 2).$

7. $ax(x + a),\ a^2(x - a),\ x^2 - a^2.$
Ans. $a^2x(x^2 - a^2).$

8. $x^2 + 3x + 2,\ x^2 + 4x + 3,\ x^2 + 5x + 6.$
Ans. $(x + 1)(x + 2)(x + 3).$

9. $x^2 - x - 30,\ x^2 - 11x + 30,\ x^2 - 25.$
Ans. $(x - 6)(x + 5)(x - 5).$

10. $6x^2 + 37x + 56,\ 8x^2 + 38x + 35,\ 12x^2 + 47x + 40.$
Ans. $(2x + 7)(3x + 8)(4x + 5).$

11. $5(x^2 - x + 1),\ 6(x^2 + 1),\ 7(x^3 + 1).$
Ans. $210(x^2 + 1)(x^3 + 1).$

12. $x^4 + a^2x^2 + a^4,\ a^2x^2 + a^3x + a^4,\ ax^2 - a^2x + a^3.$
Ans. $a^2(x^4 + a^2x^2 + a^4).$

13. $x^2 + (a + b)x + ab,\ \ x^2 + (a + c)x + ac,$
$x^2 + (b + c)x + bc.$ *Ans.* $(x+a)(x+b)(x+c).$

14. $1 - x,\ 1 + x,\ 1 + x^2,\ 1 + x^4,\ 1 + x^8.$
Ans. $1 - x^{16}.$

CHAPTER V.

FRACTIONS.

114. A **Fraction** is a quotient expressed by placing the dividend over the divisor, with a line between them. Thus, $\frac{2}{3}$ and $\frac{a}{b}$ are fractions.

115. The **Numerator** of a fraction is the expression above the line.

116. The **Denominator** of a fraction is the expression below the line.

117. The **Terms** of a fraction are its numerator and denominator.

118. A fraction is in its **Lowest Terms** when its terms have no common factor.

119. An **Entire Quantity** is a quantity, no part of which is expressed under a fractional form. Thus, $a+b-c$ is an entire quantity.

An entire quantity may be considered as a fraction having 1 for its denominator. Thus, $a=\frac{a}{1}$.

120. A **Mixed Quantity** is a quantity which contains an entire part and a part expressed under a fractional form. Thus, $a+\frac{b}{c}$ is a mixed quantity.

121. A **Simple Fraction** is a fraction whose terms are entire. Thus, $\frac{a-b}{c+d}$ is a simple fraction.

122. **A Complex Fraction** is a fraction which has a fraction in one or both of its terms. Thus,

$$\frac{a-\frac{b}{c}}{d+e} \text{ and } \frac{a+\frac{b}{c}}{d+\frac{e}{f}} \text{ are complex fractions.}$$

123. The **Reciprocal** of a quantity is the quotient obtained by dividing 1 by that quantity. Thus, the reciprocal of a is $\frac{1}{a}$.

124. *Each sign* in the numerator and denominator of a fraction affects only the quantity to which it is prefixed; but the sign prefixed to the dividing line affects the fraction as a *whole*. Thus, in the fraction $-\frac{a-b}{c-d}$, the sign of a is $+$, that of b is $-$, that of c is $+$, and that of d is $-$; but the sign $-$ prefixed to the dividing line is the sign of the fraction.

125. The **Apparent Sign** of a fraction is the sign prefixed to the dividing line of that fraction. Thus, the apparent sign of $\frac{a-b}{c}$ is $+$.

126. The **Real Sign** of a fraction is the sign of its numerical value. Thus, the real sign of $\frac{a-b}{c}$ is $-$, if $a=6$, $b=8$, and $c=5$.

127. Th. *If the apparent sign of a fraction, or the sign of each term of the numerator, or the sign of each term of the denominator, be changed, the real sign of the fraction will be changed.*

Thus, $\frac{ab}{b}=a$, but $-\frac{ab}{b}=-a$;

$$\frac{ab - bc}{b} = a - c, \quad \text{but} \quad \frac{-ab + bc}{b} = -a + c;$$

and $$\frac{a^2 - b^2}{a - b} = a + b, \quad \text{but} \quad \frac{a^2 - b^2}{-a + b} = -a - b.$$

COR. *If any two of these changes be made at the same time, the real sign of the fraction will not be changed.*

Thus, $\frac{ab}{b} = a$, and $\frac{-ab}{-b} = a$.

REDUCTION.

128. TH. *The value of a fraction is not changed by multiplying or dividing both of its terms by the same quantity.*

The fraction $\frac{a}{b}$ is the quotient obtained by dividing a by b (**114**);

$$\therefore \quad \frac{a}{b} \times b = a;$$

for the product of the quotient and divisor is equal to the dividend.

Multiplying both members of this equation by n,

$$\frac{a}{b} \times bn = an \quad (\mathbf{64}, 3).$$

Dividing both members of this equation by bn,

$$\frac{a}{b} = \frac{an}{bn}, \quad \text{or} \quad \frac{an}{bn} = \frac{a}{b} \quad (\mathbf{64}, 6).$$

129. To reduce a fraction to its lowest terms.

1. Let it be required to reduce $\frac{10acx^2}{15bcx^3}$ to its lowest terms.

The g. c. d. of the terms of this fraction is $5cx^2$. Dividing both terms of the fraction by this, the result is $\frac{2a}{3bx}$;

$$\therefore \quad \frac{10acx^2}{15bcx^3} = \frac{2a}{3bx} \ \textbf{(128)}.$$

2. Let it be required to reduce $\frac{3a^2 - 3ab}{3a^2 + 3ab}$ to its lowest terms.

$$\frac{3a^2 - 3ab}{3a^2 + 3ab} = \frac{3a\,(a - b)}{3a\,(a + b)} = \frac{a - b}{a + b}.$$

RULES.

I. *Divide both terms of the given fraction by their g. c. d.*

Or, II. *Resolve both terms of the given fraction into their prime factors, and then cancel those factors that are common.*

EXERCISES.

Reduce each of the following fractions to its lowest terms:

1. $\frac{a^2 + 2a - 3}{a^2 + 6a - 7}$. *Ans.* $\frac{a + 3}{a + 7}$.

2. $\frac{2a^2 - 6a - 8}{2a^2 - 8a - 10}$. *Ans.* $\frac{a - 4}{a - 5}$.

3. $\frac{a^3 - 6a^2 + 11a - 6}{a^2 - 3a + 2}$. *Ans.* $a - 3$.

4. $\frac{14a^2 - 7ab}{10ac - 5bc}$. *Ans.* $\frac{7a}{5c}$.

5. $\frac{x^2 + 11x + 30}{9x^3 + 53x^2 - 9x - 18}$. *Ans.* $\frac{x + 5}{9x^2 - x - 3}$.

6. $\frac{(a - x)^2}{a^3 - x^3}$. *Ans.* $\frac{a - x}{a^2 + ax + x^2}$.

7. $\frac{x^4 - 4x^3 + 4x^2 - 9}{x^4 + 2x^2 + 9}$. *Ans.* $\frac{x^2 - 2x - 3}{x^2 + 2x + 3}$.

8. $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}$. *Ans.* $\dfrac{a+b+c}{a-b-c}$.

9. $\dfrac{21a^3b^2c-9ab^3c^2}{15a^2b^2c-3a^5b^4c^2-12ab^2c}$. *Ans.* $\dfrac{7a^2-3bc}{5a-a^4b^2c-4}$.

10. $\dfrac{12a^3x^4+2a^2x^5}{18ab^2x+3b^2x^2}$. *Ans.* $\dfrac{2a^2x^3}{3b^2}$.

11. $\dfrac{a^3-x^3}{(a-x)^2}$. *Ans.* $\dfrac{a^2+ax+x^2}{a-x}$.

12. $\dfrac{a^3+(1+a)\,ay+y^2}{a^4-y^2}$. *Ans.* $\dfrac{a+y}{a^2-y}$.

13. $\dfrac{6ac+10bc+9ad+15bd}{6c^2+9cd-2c-3d}$. *Ans.* $\dfrac{3a+5b}{3c-1}$.

14. $\dfrac{x^2+2x-3}{x^2+5x-6}$. *Ans.* $\dfrac{x+3}{x+6}$.

15. $\dfrac{9x^3+53x^2-9x-18}{5x^2+55x+150}$. *Ans.* $\dfrac{9x^2-x-3}{5x+25}$.

16. $\dfrac{2x^3+x^2-8x+5}{x^3+x^2-2x}$. *Ans.* $\dfrac{2x^2+3x-5}{x^2+2x}$.

17. $\dfrac{a^2-3ab+ac+2b^2-2bc}{a^2-b^2+2bc-c^2}$. *Ans.* $\dfrac{a-2b}{a+b-c}$.

18. $\dfrac{x^4-8x^3+6x^2+8x-7}{x^4+6x^3-8x^2-6x+7}$. *Ans.* $\dfrac{x-7}{x+7}$.

19. $\dfrac{x^3-2x^2+x-12}{x^2+2x-15}$. *Ans.* $\dfrac{x^2+x+4}{x+5}$.

20. $\dfrac{x^2-5x+4}{x^2+2x-24}$. *Ans.* $\dfrac{x-1}{x+6}$.

21. $\dfrac{x^3-3x+2}{x^3+4x^2-5}$. *Ans.* $\dfrac{x^2+x-2}{x^2+5x+5}$.

22. $\dfrac{6x^2+29x+35}{14x^2+39x+10}$. *Ans.* $\dfrac{3x+7}{7x+2}$.

23. $\dfrac{2x^3+7x-9}{5x^3-3x^2-4x+2}$. *Ans.* $\dfrac{2x^2+2x+9}{5x^2+2x-2}$.

130. To reduce a fraction to an entire or mixed quantity.

1. Let it be required to reduce $\frac{a^2 - ab + ac}{a}$ to an entire quantity.

$$\frac{a^2 - ab + ac}{a} = a - b + c \ (87).$$

2. Let it be required to reduce $\frac{a^2 - ab + bc}{a}$ to a mixed quantity.

$$\frac{a^2 - ab + bc}{a} = a - b + \frac{bc}{a}.$$

RULE.

Divide the numerator of the given fraction by its denominator, expressing any term of the quotient under a fractional form, if necessary.

EXERCISES.

Reduce each of the following fractions to a mixed quantity:

1. $\frac{3x + a}{3}$. *Ans.* $x + \frac{a}{3}$.

2. $\frac{3ax + 2by + c}{2b}$. *Ans.* $\frac{3ax}{2b} + y + \frac{c}{2b}$.

3. $\frac{bx - 3a}{b}$. *Ans.* $x - \frac{3a}{b}$.

4. $\frac{(a - b)^2}{a}$. *Ans.* $a - 2b + \frac{b^2}{a}$.

5. $\frac{(a - b)^3}{b}$. *Ans.* $\frac{a^3}{b} - 3a^2 + 3ab - b^2$.

6. $-\frac{7a^3 - 3a^2b + 2b^3}{-b}$. *Ans.* $\frac{7a^3}{b} - 3a^2 + 2b^2$.

7. $\frac{8b^2x - 3a^2 + 2ab}{x}$. *Ans.* $8b^2 - \frac{3a^2 - 2ab}{x}$.

131. To reduce an entire quantity to a fraction which shall have a given denominator.

Let it be required to reduce $a - b$ to a fraction whose denominator shall be $a + b$.

$$a - b = \frac{a - b}{1} = \frac{(a - b)(a + b)}{a + b} = \frac{a^2 - b^2}{a + b}.$$

RULE.

Consider the given entire quantity as a fraction whose denominator is unity; then multiply both terms of this fraction by the given denominator.

EXERCISES.

1. Reduce $x - 3$ to a fraction whose denominator shall be $x^2 - 3x + 2$. *Ans.* $\dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 3x + 2}$.

2. Reduce $a + b$ to a fraction whose denominator shall be $a^2 + 2ab + b^2$. *Ans.* $\dfrac{a^3 + 3a^2b + 3ab^2 + b^3}{a^2 + 2ab + b^2}$.

3. Reduce $x + 1$ to a fraction whose denominator shall be $x^3 + 9x^2 + 26x + 24$. *Ans.* $\dfrac{x^4 + 10x^3 + 35x^2 + 50x + 24}{x^3 + 9x^2 + 26x + 24}$.

4. Reduce $x^2 - ax + a^2$ to a fraction whose denominator shall be $x + a$. *Ans.* $\dfrac{x^3 + a^3}{x + a}$.

5. Reduce $x^2 + x + 1$ to a fraction whose denominator shall be $x - 1$. *Ans.* $\dfrac{x^3 - 1}{x - 1}$.

6. Reduce $x^n + a^n$ to a fraction whose denominator shall be $x^n - a^n$. *Ans* $\dfrac{x^{2n} - a^{2n}}{x^n - a^n}$.

132. To reduce one fraction to another which shall have a given denominator.

Let it be required to reduce $\frac{a-b}{a+b}$ to an equivalent fraction whose denominator shall be a^2-b^2.

Dividing a^2-b^2 by $a+b$, the quotient is found to be $a-b$. Multiplying both terms of $\frac{a-b}{a+b}$ by this quotient, the result is $\frac{(a-b)^2}{a^2-b^2}$;

$$\therefore \quad \frac{a-b}{a+b} = \frac{(a-b)^2}{a^2-b^2}.$$

RULE.

Divide the denominator of the required fraction by that of the given fraction; then multiply both terms of the given fraction by the quotient.

EXERCISES.

1. Reduce $\frac{x+3}{x+7}$ to a fraction whose denominator shall be x^2+6x-7. *Ans.* $\frac{x^2+2x-3}{x^2+6x-7}$.

2. Reduce $\frac{x-4}{x-5}$ to a fraction whose denominator shall be x^2-4x-5. *Ans.* $\frac{x^2-3x-4}{x^2-4x-5}$.

3. Reduce $\frac{3x-1}{2x-1}$ to a fraction whose denominator shall be $2x^3-11x^2+17x-6$. *Ans.* $\frac{3x^3-16x^2+23x-6}{2x^3-11x^2+17x-6}$.

4. Reduce $\frac{x+a}{x+b}$ to a fraction whose denominator shall be $x^2+(b+c)x+bc$. *Ans.* $\frac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}$.

133. To reduce a mixed quantity to a fraction.

Let it be required to reduce $a - b + \frac{bc}{a}$ to a fraction.

$$a - b + \frac{bc}{a} = \frac{a^2 - ab + bc}{a};$$

for, if $\frac{a^2 - ab + bc}{a}$ be reduced to a mixed quantity, the result will be $a - b + \frac{bc}{a}$ **(130)**.

RULE.

Multiply the entire part of the given expression by the denominator of the fractional part; add the numerator to the product, or subtract it from the product, according as the apparent sign of the fractional part is + or —; then under the result draw a line, and under the line write the given denominator.

EXERCISES.

Reduce each of the following expressions to a fraction:

1. $x - \frac{3a}{b}$. *Ans.* $\frac{bx - 3a}{b}$.
2. $5 - \frac{7 - 2a}{3a}$. *Ans.* $\frac{17a - 7}{3a}$.
3. $1 - \frac{b - a - 1}{a}$. *Ans.* $\frac{2a - b + 1}{a}$.
4. $2x + 1 + \frac{3 - x}{5x}$. *Ans.* $\frac{10x^2 + 4x + 3}{5x}$.
5. $3 - \frac{3}{1 - x^2}$. *Ans.* $\frac{3x^2}{x^2 - 1}$.
6. $ax + 5 - \frac{ax^2 - b}{x + b}$. *Ans.* $\frac{abx + 5x + 6b}{x + b}$.
7. $x^2 - ax + a^2 - \frac{-x^3}{x + a}$. *Ans.* $\frac{2x^3 + a^3}{x + a}$.

134. To reduce fractions having unequal denominators to others which shall have a common denominator.

1. Let it be required to reduce $\frac{a}{b}$ and $\frac{c}{d}$ to other fractions which shall have a common denominator.

If both terms of each of these fractions be multiplied by the denominator of the other, the values of the fractions will not be changed **(128)**, and the denominators of the new fractions will be equal, because each will be the product of the given denominators. Thus,

$$\frac{a}{b} = \frac{ad}{bd}, \quad \text{and} \quad \frac{c}{d} = \frac{bc}{bd}.$$

2. Let it be required to reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ to other fractions which shall have a common denominator.

If both terms of each of these fractions be multiplied by the product of all the denominators except its own, the values of the fractions will not be changed, and the denominators of the new fractions will be equal, because each will be equal to the product of all the given denominators. Thus,

$$\frac{a}{b} = \frac{adf}{bdf}, \quad \frac{c}{d} = \frac{bcf}{bdf}, \quad \text{and} \quad \frac{e}{f} = \frac{bde}{bdf}.$$

RULES.

I. *If there are only two given fractions, multiply both terms of each by the denominator of the other.*

II. *If the number of given fractions is greater than two, multiply both terms of each by the product of the denominators of the other fractions.*

EXERCISES.

Reduce each of the following groups of fractions to another which shall have a common denominator:

1. $\frac{2}{a}, \frac{3}{b}.$ *Ans.* $\frac{2b}{ab}, \frac{3a}{ab}.$

2. $\frac{1}{a}, \frac{2}{b}, \frac{3}{c}.$ *Ans.* $\frac{bc}{abc}, \frac{2ac}{abc}, \frac{3ab}{abc}.$

3. $\frac{a}{x}, \frac{b}{x-1}, \frac{c}{x+1}.$

Ans. $\frac{a(x^2-1)}{x^3-x}, \frac{b(x^2+x)}{x^3-x}, \frac{c(x^2-x)}{x^3-x}.$

4. $\frac{1}{3}, \frac{a}{2}, -\frac{x^2}{a-x}.$

Ans. $\frac{2(a-x)}{6(a-x)}, \frac{3a(a-x)}{6(a-x)}, -\frac{6x^2}{6(a-x)}.$

5. $\frac{a-b}{a}, \frac{ax}{a-b}, \frac{b}{c}.$

Ans. $\frac{c(a-b)^2}{ac(a-b)}, \frac{a^2cx}{ac(a-b)}, \frac{ab(a-b)}{ac(a-b)}.$

6. $\frac{1}{a-x}, \frac{1}{a+x}.$ *Ans.* $\frac{a+x}{a^2-x^2}, \frac{a-x}{a^2-x^2}.$

7. $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}.$ *Ans.* $\frac{bcx}{abc}, \frac{acy}{abc}, \frac{abz}{abc}.$

8. $\frac{x^2}{2ab}, \frac{y^2}{3ac}, \frac{z^2}{4bc}.$

Ans. $\frac{12abc^2x^2}{24a^2b^2c^2}, \frac{8ab^2cy^2}{24a^2b^2c^2}, \frac{6a^2bcz^2}{24a^2b^2c^2}.$

9. $\frac{a}{1-x}, \frac{b}{1+x}, \frac{c-1}{x}.$

Ans. $\frac{a(x+x^2)}{x-x^3}, \frac{b(x-x^2)}{x-x^3}, \frac{(c-1)(1-x^2)}{x-x^3}.$

135. To reduce fractions having unequal denominators to others which shall have the least common denominator.

Let it be required to reduce $\frac{a}{4bc}$, $\frac{d}{8b^3}$, and $\frac{e}{12bc^2f}$ to other fractions which shall have the least common denominator.

The l. c. m. of the denominators is $24b^3c^2f$. Dividing this by each denominator, the quotients are found to be

$$6b^2cf, \quad 3c^2f, \quad \text{and} \quad 2b^2.$$

Multiplying both terms of $\frac{a}{4bc}$ by $6b^2cf$, both terms of $\frac{d}{8b^3}$ by $3c^2f$, and both terms of $\frac{e}{12bc^2f}$ by $2b^2$, the resulting fractions are

$$\frac{6ab^2cf}{24b^3c^2f}, \quad \frac{3c^2df}{24b^3c^2f}, \quad \text{and} \quad \frac{2b^2e}{24b^3c^2f}.$$

These have the least common denominator, for $24b^3c^2f$ is the l. c. m. of the given denominators.

RULE.

Divide the l. c. m. of all the given denominators by each denominator separately; then multiply both terms of each fraction by the corresponding quotient.

SCH. 1. Before commencing the operation, each fraction must be in its lowest terms.

SCH. 2. The word *least* here has no reference to *numerical value.* It refers simply to the number of prime factors in the common denominator.

EXERCISES.

Reduce each of the following groups of fractions to another which shall have the least common denominator:

1. $\frac{a}{mx}$, $\frac{b}{my}$, $\frac{c}{mz}$. *Ans.* $\frac{ayz}{mxyz}$, $\frac{bxz}{mxyz}$, $\frac{cxy}{mxyz}$.

2. $\frac{a}{a-x}$, $\frac{b}{a+x}$, $\frac{c}{a^2-x^2}$.

Ans. $\frac{a(a+x)}{a^2-x^2}$, $\frac{b(a-x)}{a^2-x^2}$, $\frac{c}{a^2-x^2}$.

3. $\frac{a}{x-1}$, $\frac{b}{x^2-1}$, $\frac{c}{x^2-2x+1}$.

Ans. $\frac{a(x^2-1)}{(x+1)(x-1)^2}$, $\frac{b(x-1)}{(x+1)(x-1)^2}$, $\frac{c(x+1)}{(x+1)(x-1)^2}$.

4. $\frac{x-1}{x+1}$, $\frac{x^2+1}{x^2-x+1}$, $\frac{x^3}{x^3+1}$.

Ans. $\frac{(x-1)(x^2-x+1)}{x^3+1}$, $\frac{(x^2+1)(x+1)}{x^3+1}$, $\frac{x^3}{x^3+1}$.

ADDITION.

136. To find the sum of given fractions.

1. Let it be required to find the sum of $\frac{a}{b}$, $\frac{c}{b}$, and $\frac{d}{b}$.

Since the product of the quotient and divisor is equal to the dividend,

$$\frac{a}{b}\times b+\frac{c}{b}\times b+\frac{d}{b}\times b=a+c+d.$$

Dividing both members of this equation by b,

$$\frac{a}{b}+\frac{c}{b}+\frac{d}{b}=\frac{a+c+d}{b}.$$

2. Let it be required to find the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

$$\frac{a}{b}+\frac{c}{d}+\frac{e}{f}=\frac{adf}{bdf}+\frac{bcf}{bdf}+\frac{bde}{bdf}=\frac{adf+bcf+bde}{bdf}.$$

RULES.

I. *If the given fractions have a common denominator, form a fraction whose numerator is the sum of the given numerators, and whose denominator is the given common denominator. The result will be the sum of the given fractions.*

II. *If the given fractions have not a common denominator, reduce them to others which shall have a common denominator; then proceed with the results as directed in* I.

EXERCISES.

Find the sum of

1. $\frac{2}{x}$ and $\frac{3}{y}$. *Ans.* $\frac{2y+3x}{xy}$.

2. $\frac{a}{x}$, $\frac{b}{x-1}$, and $\frac{c}{x+1}$.
Ans. $\frac{a(x^2-1)+b(x^2+x)+c(x^2-x)}{x^3-x}$.

3. $\frac{1}{3}$, $\frac{a}{2}$, and $\frac{x^2}{x-a}$.
Ans. $\frac{2(x-a)+3a(x-a)+6x^2}{6(x-a)}$.

4. $\frac{a-b}{a}$, $\frac{ax}{a-b}$, and $\frac{b}{c}$.
Ans. $\frac{c(a-b)^2+a^2cx+ab(a-b)}{ac(a-b)}$.

5. $\frac{a}{mx}$, $\frac{b}{my}$, and $\frac{c}{mz}$. *Ans.* $\frac{ayz+bxz+cxy}{mxyz}$.

6. $x-\frac{2-x}{3}$ and $5x-\frac{3-2x}{4}$.
Ans. $6x+\frac{10x-17}{12}$.

7. $\frac{a}{a+b}$, $\frac{b}{a-b}$, and $\frac{a^2+b^2}{a^2-b^2}$. *Ans.* $\frac{2(a^2+b^2)}{a^2-b^2}$.

8. $\frac{a}{a-x}$, $\frac{3a}{a+x}$, and $\frac{2ax}{x^2-a^2}$. *Ans.* $\frac{4a}{a+x}$.

9. $\frac{1+x}{1+x+x^2}$ and $\frac{1-x}{1-x+x^2}$. *Ans.* $\frac{2}{1+x^2+x^4}$.

10. 2, $\frac{a^2+b^2}{a^2-b^2}$, and $-\frac{a-b}{a+b}$. *Ans.* $\frac{2a^2+2ab-2b^2}{a^2-b^2}$.

SUBTRACTION.

137. To find the difference between two fractions.

1. Let it be required to subtract $\frac{c}{b}$ from $\frac{a}{b}$.

$$\frac{a}{b}\times b-\frac{c}{b}\times b=a-c.$$

Dividing both members of this equation by b,

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}.$$

2. Let it be required to subtract $\frac{c}{d}$ from $\frac{a}{b}$.

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}.$$

RULES.

I. *If the given fractions have a common denominator, form a fraction whose numerator is the remainder obtained by subtracting the numerator of the subtrahend from that of the minuend, and whose denominator is the given common denominator. The result will be the difference required.*

II. *If the given fractions have not a common denominator, reduce them to others which shall have a common denominator; then proceed with the results as directed in* I.

EXERCISES.

Subtract

1. $\dfrac{a}{a+b}$ from $\dfrac{a}{a-b}$. *Ans.* $\dfrac{2ab}{a^2-b^2}$.

2. $\dfrac{b}{2a-2b}$ from $\dfrac{a}{2a-2b}$. *Ans.* $\dfrac{1}{2}$.

3. $x+\dfrac{a-x}{c}$ from $3x+\dfrac{x}{b}$. *Ans.* $2x+\dfrac{bx+cx-ab}{bc}$.

4. $\dfrac{a}{a^2+b^2}$ from $\dfrac{1}{a+b}+\dfrac{b}{a^2-b^2}$. *Ans.* $\dfrac{2ab^2}{a^4-b^4}$.

5. $\dfrac{a+b}{m}-\dfrac{a-b}{n}$ from $\dfrac{n(a+b)+m(a+b)}{mn}$.

Ans. $\dfrac{2a}{n}$.

6. $\dfrac{7}{1+2x}+\dfrac{4-20x}{4x^2-1}$ from $\dfrac{3}{1-2x}$. *Ans.* 0.

7. $\dfrac{2a-b-c}{3}$ from $\dfrac{3a-4b}{2}+\dfrac{15a-4c}{12}$.

Ans. $\dfrac{25a-20b}{12}$.

MULTIPLICATION.

138. To find the product of given fractions.

Let it be required to find the product of $\dfrac{a}{b}$ and $\dfrac{c}{d}$.

$$\frac{a}{b}\times b=a,$$

and
$$\frac{c}{d}\times d=c;$$

$$\therefore\ \frac{a}{b}\times\frac{c}{d}\times bd=ac\ (\mathbf{64},\ 3).$$

Dividing both members of this equation by bd,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

RULE.

Divide the product of the numerators of the given fractions by the product of their denominators.

COR. 1. Since an entire quantity may be considered as a fraction having 1 for its denominator, and a mixed quantity may be reduced to a fraction, it follows that the rule may be made to embrace every case in which a fraction is a factor. Thus,

$$a \times \frac{c}{d} = \frac{a}{1} \times \frac{c}{d} = \frac{ac}{d},$$

and $$\left(a + \frac{c}{d}\right) \times \frac{e}{f} = \frac{ad + c}{d} \times \frac{e}{f} = \frac{ade + ce}{df}.$$

COR. 2. The product of a quantity and its reciprocal is 1.

Thus, $$a \times \frac{1}{a} = \frac{a}{1} \times \frac{1}{a} = \frac{a}{a} = 1.$$

SCH. When any numerator and any denominator contain a common factor, the operation of multiplication may be abridged by cancellation.

EXERCISES.

Find the product of

1. $a - \frac{ax}{b}$ and $\frac{c}{d}$. *Ans.* $\frac{abc - acx}{bd}$.

2. $\frac{(a - b)^2}{a + b}$ and $\frac{b}{x(a - b)}$. *Ans.* $\frac{(a - b)\,b}{(a + b)\,x}$.

3. $\frac{2a}{x}$, $\frac{3b}{c}$, and $-\frac{4ac}{d}$. *Ans.* $-\frac{24a^2b}{dx}$.

4. $\dfrac{a^2+ab}{a^2+b^2}$ and $\dfrac{a^3-b^3}{ab(a+b)}$. *Ans.* $\dfrac{a^3-b^3}{b(a^2+b^2)}$.

5. $\dfrac{x(a-x)}{(a+x)^2}$ and $\dfrac{a(a+x)}{(a-x)^2}$. *Ans.* $\dfrac{ax}{a^2-x^2}$.

6. $\dfrac{a^4-b^4}{(a-b)^2}$ and $\dfrac{a-b}{a^2+ab}$. *Ans.* $a+\dfrac{b^2}{a}$.

7. $\dfrac{2ax}{3by}$, $\dfrac{(a+x)(a-x)}{c^2-x^2}$, $\dfrac{bc+bx}{a^2+ax}$, and $\dfrac{c-x}{a-x}$.

Ans. $\dfrac{2x}{3y}$.

8. $\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}-\dfrac{4a^2}{x^2-a^2}$ and $\dfrac{x+a}{2a}$. *Ans.* 2.

9. $\dfrac{a}{x}+\dfrac{b}{y}$ and $\dfrac{x}{b}+\dfrac{y}{a}$.

Ans. $\dfrac{ab(x^2+y^2)+(a^2+b^2)xy}{abxy}$.

10. $\dfrac{1}{a}+\dfrac{1}{x}$ and $\dfrac{1}{a^2}-\dfrac{1}{x^2}$. *Ans.* $\dfrac{x^3+ax^2-a^2x-a^3}{a^3x^3}$.

11. $\dfrac{1}{(a-b)^2}-\dfrac{1}{c^2}$ and $\dfrac{a^2-b^2}{a-b-c}$

Ans. $\dfrac{b^2-a^2-ac-bc}{ac^2-bc^2}$.

12. $\dfrac{a^2+b^2}{a^2-b^2}$ and $\dfrac{a+b}{a-b}$. *Ans.* $\dfrac{a^2+b^2}{(a-b)^2}$.

13. $\dfrac{x^3+y^3}{x^2-y^2}$ and $\dfrac{x-y}{x^2-xy+y^2}$. *Ans.* 1.

14. $1+\dfrac{1}{x}$, $\left(1-\dfrac{1}{x}\right)^2$, and $\dfrac{x}{x^2-1}$. *Ans.* $\dfrac{x-1}{x^2}$.

15. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2}$, $\dfrac{3}{2ab-2b^2}$, and $\dfrac{a^2+ab}{a-b}$.

Ans. $\dfrac{3a}{2b}$.

DIVISION.

139. To divide an expression by a fraction.

Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$.

$$1 = \frac{c}{d} \times \frac{d}{c} \text{ (138)}.$$

Multiplying both members of this equation by $\frac{a}{b}$,

$$\frac{a}{b} = \frac{c}{d} \times \frac{a}{b} \times \frac{d}{c}.$$

Dividing both members of this equation by $\frac{c}{d}$,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

RULE.

Multiply the dividend by the divisor inverted.

COR. 1. The reciprocal of a fraction is that fraction inverted. Thus,

the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, for $1 \div \frac{a}{b} = \frac{b}{a}$.

Hence, the rule for dividing an expression by a fraction may be stated as follows:

Multiply the dividend by the reciprocal of the divisor.

COR. 2. Dividing by any quantity is the same in effect as multiplying by its reciprocal, and multiplying by a quantity is the same in effect as dividing by its reciprocal. Thus,

$$a \div b = a \times \frac{1}{b}, \quad \text{and} \quad a \times b = a \div \frac{1}{b}.$$

EXERCISES.

Divide

1. $\frac{5x^2}{7}$ by $\frac{x}{3}$. *Ans.* $\frac{15x}{7}$.

2. $\frac{x-1}{6}$ by $\frac{2x}{3}$. *Ans.* $\frac{x-1}{4x}$.

3. $\frac{x}{x+1}$ by $\frac{x}{2}$. *Ans.* $\frac{2}{x+1}$.

4. $\frac{x-a}{4cd^2}$ by $\frac{3cx}{5d}$. *Ans.* $\frac{5(x-a)}{12c^2dx}$.

5. $\frac{ax-x^2}{(a+x)^2}$ by $\frac{x^2}{a^2-x^2}$. *Ans.* $\frac{(a-x)^2}{x(a+x)}$.

6. $\frac{4(x^2-ax)}{a(x+a)}$ by $\frac{6ax}{x^2-a^2}$. *Ans.* $\frac{2(x-a)^2}{3a^2}$.

7. $\frac{2a^2}{x^3+a^3}$ by $\frac{a}{x+a}$. *Ans.* $\frac{2a}{x^2-ax+a^2}$.

8. $\frac{x+1}{x-1}$ by $\frac{x+1}{1-x^2}$. *Ans.* $-(x+1)$.

9. $\frac{x+y}{3}-\frac{y}{2}$ by $\frac{x}{a}-\frac{y}{2a}$. *Ans.* $\frac{a}{3}$.

10. $\frac{x}{a}-\frac{a}{x}$ by $\frac{1}{a}+\frac{1}{x}$. *Ans.* $x-a$.

11. $a^2-(b-c)^2$ by $\frac{a+b-c}{a+b+c}$. *Ans.* $(a+c)^2-b^2$.

12. x^2-x+1 by $\frac{1}{x^2}+\frac{1}{x}+1$. *Ans.* $\frac{x^2(x^2-x+1)}{x^2+x+1}$.

13. $\frac{x^2+xy}{x-y}$ by $\frac{x^4-y^4}{(x-y)^2}$. *Ans.* $\frac{x}{x^2+y^2}$.

14. $\frac{x^3+y^3}{x^2-y^2}$ by $\frac{x^2-xy+y^2}{x-y}$. *Ans.* 1.

15. $\left(1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)^2$ by $x-\frac{1}{x}$. *Ans.* $\frac{x-1}{x^2}$.

16. $\dfrac{x^4-b^4}{x^2-2bx+b^2}\times\dfrac{x^5-b^2x^3}{x^3+b^3}$ by $\dfrac{x^4-2bx^3+b^2x^2}{x^2-bx+b^2}\times\dfrac{x^2+bx}{x-b}$. *Ans.* $\dfrac{x^2+b^2}{x-b}$.

17. $2\frac{1}{4}-\frac{1}{3}x$ by $\frac{2}{3}x-1\frac{1}{2}$. *Ans.* $\dfrac{27-4x}{2(4x-9)}$.

18. $2\frac{1}{3}-\frac{1}{2}(x-2)$ by $\frac{1}{3}(x+1)-4\frac{1}{2}$. *Ans.* $\dfrac{20-3x}{2x-25}$.

19. $\dfrac{1}{1+x}$ by $1-\dfrac{1}{1+x}$. *Ans.* $\dfrac{1}{x}$.

20. $\dfrac{1}{1+x}+\dfrac{x}{1-x}$ by $\dfrac{1}{1-x}-\dfrac{x}{1+x}$. *Ans.* 1.

21. $\dfrac{a^2+b^2}{2a^2}-\dfrac{2b^2}{a^2+b^2}$ by $\dfrac{a^2+b^2}{2b^2}-\dfrac{2a^2}{a^2+b^2}$. *Ans.* $\dfrac{b^2}{a^2}$.

22. $\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}$ by $\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}$. *Ans.* $\dfrac{a^2+x^2}{2ax}$.

SIMPLIFICATION OF COMPLEX FRACTIONS.

140. The methods used in the simplification of complex fractions are illustrated in the following

EXERCISES.

1. Simplify $\dfrac{1}{a+\dfrac{b}{c}}$.

This fraction may be simplified by multiplying both of its terms by c, the denominator of the simple fraction. Thus,

$$\frac{1}{a+\frac{b}{c}}=\frac{1\times c}{\left(a+\frac{b}{c}\right)c}=\frac{c}{ac+b}.$$

The same result may be obtained as follows:

$$\frac{1}{a+\frac{b}{c}}=\frac{1}{\frac{ac+b}{c}}=1\div\frac{ac+b}{c}=\frac{c}{ac+b}.$$

2. Simplify $\dfrac{a+\frac{b}{x}}{c-\frac{d}{y}}$.

This fraction may be simplified by multiplying both of its terms by xy, the l. c. m. of the denominators of the fractions $\frac{b}{x}$ and $\frac{d}{y}$. Thus,

$$\frac{a+\frac{b}{x}}{c-\frac{d}{y}}=\frac{\left(a+\frac{b}{x}\right)xy}{\left(c-\frac{d}{y}\right)xy}=\frac{axy+by}{cxy-dx}=\frac{(ax+b)\,y}{x\,(cy-d)}.$$

The same result may be obtained as follows:

$$\frac{a+\frac{b}{x}}{c-\frac{d}{y}}=\frac{\frac{ax+b}{x}}{\frac{cy-d}{y}}=\frac{ax+b}{x}\div\frac{cy-d}{y}=\frac{(ax+b)\,y}{x\,(cy-d)}.$$

Simplify:

3. $\dfrac{1}{a-\frac{1}{b}}$. *Ans.* $\dfrac{b}{ab-1}$.

4. $\dfrac{a}{c-\frac{d}{x}}$. *Ans.* $\dfrac{ax}{cx-d}$.

5. $\dfrac{a + \dfrac{b}{x}}{c}$. *Ans.* $\dfrac{ax + b}{cx}$.

6. $\dfrac{\dfrac{1}{x} + \dfrac{1}{y}}{\dfrac{1}{x} - \dfrac{1}{y}}$. *Ans.* $\dfrac{y + x}{y - x}$.

7. $\dfrac{1 - \dfrac{a^3}{x^3}}{\dfrac{1}{x^2} - \dfrac{a}{x^3}}$. *Ans.* $x^2 + ax + a^2$.

8. $\dfrac{1}{a - \dfrac{1}{b - \dfrac{1}{c}}}$. *Ans.* $\dfrac{bc - 1}{abc - a - c}$.

9. $\dfrac{z}{x - \dfrac{xy}{z + y}}$. *Ans.* $\dfrac{y + z}{x}$.

10. $\dfrac{a + \dfrac{x - a}{1 + ax}}{1 - a \times \dfrac{x - a}{1 + ax}}$. *Ans.* x.

11. $\dfrac{x - \dfrac{1}{x}}{x - 1 - \dfrac{x - 1}{1 - \dfrac{x}{3 + x}}}$. *Ans.* $-\dfrac{3(x + 1)}{x^2}$.

12. $\dfrac{a + \dfrac{b}{x + \dfrac{c}{z}}}{c + \dfrac{d}{y + \dfrac{f}{g}}}$. *Ans.* $\dfrac{(axz + ac + bz)(gy + f)}{(cgy + cf + dg)(xz + c)}$.

13. $\dfrac{a}{b + \dfrac{c}{d + \dfrac{e}{f}}}$. *Ans.* $\dfrac{adf + ae}{bdf + be + cf}$.

14. $\dfrac{a + b + \dfrac{b^2}{a}}{a + b + \dfrac{a^2}{b}}$. *Ans.* $\dfrac{b}{a}$.

15. $\dfrac{\dfrac{a}{a+b} + \dfrac{b}{a-b}}{\dfrac{a}{a-b} - \dfrac{b}{a+b}}$. *Ans.* 1.

16. $\dfrac{\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}}$. *Ans.* $\dfrac{x^2 + y^2}{2xy}$.

17. $\dfrac{\dfrac{x}{1+x} + \dfrac{1-x}{x}}{\dfrac{x}{1+x} - \dfrac{1-x}{x}}$. *Ans.* $\dfrac{1}{2x^2 - 1}$.

18. $\dfrac{\dfrac{x^2}{y^3} + \dfrac{1}{x}}{\dfrac{x}{y^2} - \dfrac{1}{y} + \dfrac{1}{x}}$. *Ans.* $\dfrac{x + y}{y}$.

19. $\dfrac{\left(a - \dfrac{b^2}{2a}\right)\left(a - \dfrac{a^2 + b^2}{a + b}\right)}{1 - \dfrac{a}{a+b}}$. *Ans.* $\dfrac{(2a^2 - b^2)(a - b)}{2a}$.

20. $\dfrac{\left(1 - \dfrac{a}{b} + \dfrac{b}{a}\right)\left(\dfrac{a+b}{2a} + \dfrac{a-b}{2b}\right)}{\left(a - 2b + \dfrac{b^2}{a}\right)\left(\dfrac{a}{a+b} + \dfrac{b}{a-b}\right)}$.

Ans. $\dfrac{(ab + b^2 - a^2)(a + b)}{2ab^2(a - b)}$.

NEGATIVE EXPONENTS.

141.

$$\frac{a^2}{a^5} = a^{2-5} = a^{-3}\ \textbf{(85)};$$

and

$$\frac{a^2}{a^5} = \frac{1}{a^3}\ \textbf{(128)};$$

$$\therefore\ a^{-3} = \frac{1}{a^3}.$$

Again,

$$\frac{1}{a^{-3}} = \frac{1}{\frac{1}{a^3}} = a^3.$$

Hence, (1) *a quantity with a negative exponent is equal to the reciprocal of the same quantity with a positive exponent*; (2) *a quantity with a positive exponent is equal to the reciprocal of the same quantity with a negative exponent*; (3) *any factor may be transferred from one term of a fraction to the other, if the sign of its exponent be changed.*

These principles are in accordance with that explained in Art. **68**, Sch. 1; for it has been shown that a positive integral exponent indicates the number of times the quantity affected by it is to be used as a factor; hence a negative integral exponent *ought* to indicate the number of times the quantity is to be used as a divisor. Thus, in the expression a^2b^3, a^2 is to be multiplied by b^3; but in the expression a^2b^{-3}, the exponent -3 indicates that b is to be used three times as a divisor, for $a^2b^{-3} = \frac{a^2}{b^3}$.

142. The rules already established for the fundamental operations are applicable to expressions containing negative exponents. Thus,

$3a^{-2}x^{-3} + 2a^{-2}x^{-3} = 5a^{-2}x^{-3}$; $7a^{-2}x^{-3} - 3a^{-2}x^{-3} = 4a^{-2}x^{-3}$;

$$a^5 \times a^{-3} = a^{5-3} = a^2, \text{ for } a^5 \times a^{-3} = a^5 \times \frac{1}{a^3} = a^2;$$

$$a^{-5} \times a^{-3} = a^{-8} = \frac{1}{a^8}, \text{ for } a^{-5} \times a^{-3} = \frac{1}{a^5} \times \frac{1}{a^3} = \frac{1}{a^8};$$

$$\frac{a^5}{a^{-3}} = a^{5-(-3)} = a^8, \text{ for } \frac{a^5}{a^{-3}} = a^5 \div \frac{1}{a^3} = a^8;$$

$$\frac{a^{-5}}{a^{-3}} = a^{-5-(-3)} = a^{-2} = \frac{1}{a^2}, \text{ for } \frac{a^{-5}}{a^{-3}} = \frac{1}{a^5} \div \frac{1}{a^3} = \frac{1}{a^2}.$$

EXERCISES.

1. Find the sum of $\frac{1}{2}a^{-2}x^{-3} + \frac{1}{3}b^{-4}y^{-2}$, $\frac{1}{5}a^{-2}x^{-3} + \frac{1}{8}b^{-4}y^{-2}$, $\frac{1}{8}a^{-2}x^{-3} + \frac{1}{6}b^{-4}y^{-2}$, and $20a^{-2}x^{-3} - b^{-4}y^{-2}$.
Ans. $20\frac{33}{40}a^{-2}x^{-3} - \frac{3}{8}b^{-4}y^{-2}$.

2. Subtract $\frac{1}{3}x^{-3} - 3x^{-2} + \frac{1}{2}x^{-1} - 1$ from $\frac{3}{4}x^{-3} + 3x^{-2} + \frac{1}{2}x^{-1} + 1$.
Ans. $\frac{5}{12}x^{-3} + 6x^{-2} + 2$.

3. Multiply $-3a^{-5} + 2a^{-4}b^{-1}$ by $-2a^{-3} - 3a^{-4}b$.
Ans. $-4a^{-7}b^{-1} + 9a^{-9}b$.

4. Divide $6ax^{-3}$ by $2ax^{-1}$. *Ans.* $3x^{-2}$.

5. Divide $a^{-3} - x^{-3}$ by $a^{-1} - x^{-1}$.
Ans. $a^{-2} + a^{-1}x^{-1} + x^{-2}$.

6. Divide $a^4 + a^2 - a^{-2} - a^{-4}$ by $a^2 - a^{-2}$.
Ans. $a^2 + 1 + a^{-2}$.

7. Show that $\dfrac{4x^2z}{3ax^{-3}} = \dfrac{4x^5z}{3a}$.

8. Show that $\dfrac{1}{axyz} = a^{-1}x^{-1}y^{-1}z^{-1}$.

CHAPTER VI.

DEFINITIONS AND GENERAL PRINCIPLES RELATING TO EQUATIONS.

143. An **Identity** is an equation whose members are either *identical,* or may be made identical by performing certain indicated operations. Thus, $2 - 3x = 2 - 3x$, $(a - x)^2 = a^2 - 2ax + x^2$, and $1 - \frac{1}{1 + x} = \frac{x}{1 + x}$ are identities.

It follows from the definition that

The members of an identity are equal for all values that may be assigned to each letter which it contains.

Thus far, the student has been almost entirely occupied with identities. Thus, the equations in Articles **80, 81, 82** are identities.

144. An **Equation of Condition** is an equation whose members are equal only for a limited number of values of each letter which it contains. Thus, $x + 1 = 9$ is an equation of condition, because its members are not equal, unless $x = 8$.

An equation of condition is called, briefly, *an equation.*

145. An **Unknown Quantity** of an equation is a letter to which a particular value or values must be given, in order that its members may become *identical.* Thus, in $x^2 - 5x = -6$, x is the unknown quantity, and the members of this equation will become identical, if 3 or 2 be substituted for x.

146. An **Unknown Term** of an equation is a term containing an unknown quantity. Thus, the unknown terms of $x^2 - 5x = -6$ are x^2 and $-5x$.

147. A **Root of an Equation** is a quantity which, when substituted for an unknown quantity in it, makes the members identical. Such a quantity is said to *satisfy* the equation. Thus, the roots of $x^2 - 5x = -6$ are 3 and 2, for $3^2 - 5 \times 3 = -6$, and $2^2 - 5 \times 2 = -6$.

148. To Solve an Equation is to find its root or its roots.

149. A **Numerical Equation** is an equation in which all the *known* quantities are represented by numbers. Thus, $2x^2 - 5x = 10x - 15$ is a numerical equation.

150. A **Literal Equation** is an equation in which the known quantities are represented entirely or in part by letters. Thus, $ax - b = cx - d$ and $ax + b = 3x + 5$ are literal equations.

151. The **Degree** of an equation, which is of such a form that no unknown quantity occurs under the *radical sign* or in a *denominator*, is denoted by the number of unknown factors in that term which contains the greatest number of such factors. Thus,

$3x - 2x + 3 = 4$ is of the first degree,
$x^2 + 5x = 6$ " " second " ,
$x^2y + x^2 - 5x = 10$ " " third " ,
$x^n + ax^{n-1} + bx^{n-2} = c$ " " nth " .

152. A **Homogeneous Equation** is an equation whose unknown terms are of the same degree. Thus,

$x^2y + y^2x = 25$ is homogeneous.

153. A **Simple Equation** is an equation of the first degree.

154. A **Quadratic Equation** is an equation of the second degree.

155. A **Cubic Equation** is an equation of the third degree.

156. **Higher Equations** are equations of higher degrees than the second.

TRANSFORMATION OF EQUATIONS.

157. To **Transform** an equation is to change its members without destroying their equality.

Clearing of Fractions and *Transposition of Terms* are the principal transformations.

Only two kinds of changes are admissible in the treatment of equations: (1) *Such as do not affect the value of either member;* (2) *Such as affect the value of both members equally.*

158. To clear an equation of fractions.

Let it be required to transform the equation

$$\frac{x}{ab} - \frac{x}{bc} + d = e$$

into another, all of whose terms shall be entire.

Multiplying both members by abc, the l. c. m. of the denominators,

$$cx - ax + abcd = abce \quad (\mathbf{64}, 3).$$

RULE.

Multiply both members of the given equation by the l. c. m. of the denominators.

COR. It is evident that an equation may be cleared of fractions by multiplying both of its members by *any* multiple of the denominators. Thus, if both members of the equation $\frac{x}{ab} - \frac{x}{bc} + d = e$ be multiplied by a^2bc, the result will be $acx - a^2x + a^2bcd = a^2bce$.

SCH. If the given equation contains a complex fraction, that fraction should be reduced to a simple one before applying the rule.

EXERCISES.

Transform each of the following equations into another, all of whose terms shall be entire:

1. $\frac{x}{2} - \frac{x}{3} + \frac{x}{12} + 2 = 4.$
Ans. $6x - 4x + x + 24 = 48.$

2. $\frac{3x^2}{4} - 12 = \frac{5x^2}{6} - 9.$
Ans. $9x^2 - 144 = 10x^2 - 108.$

3. $\frac{x^3}{2} + \frac{x^2}{3} - \frac{x}{4} = 26.$ *Ans.* $6x^3 + 4x^2 - 3x = 312.$

4. $\frac{2x}{3} + 1 = \frac{x}{12} - \frac{4}{3}.$ *Ans.* $8x + 12 = x - 16.$

5. $\frac{1}{2}x + 13\frac{3}{4} = 2x - 9\frac{3}{4}.$ *Ans.* $2x + 55 = 8x - 39.$

6. $\frac{2x - 6}{5} - \frac{x - 4}{9} = \frac{3x}{13}.$
Ans. $234x - 702 - 65x + 260 = 135x.$

7. $\frac{x}{8} - \frac{2x - 2}{2\frac{1}{2}} = \frac{3x - 4}{15} - \frac{x}{12}.$
Ans. $15x - 96x + 96 = 24x - 32 - 10x.$

8. $\frac{x - 1\frac{25}{26}}{2} - \frac{2 - 6x}{13} = y - \frac{5y - \frac{10 - 3y}{4}}{39}.$
Ans. $78x - 153 - 24 + 72x = 156y - 20y + 10 - 3y.$

159. To transpose a term from one member of an equation to the other.

Let it be required to transform the equation

$$ax - b = cx - d \qquad (1)$$

into another in which all the unknown terms shall be in the first member, and all the known terms in the second.

Subtracting cx from both members of (1),

$$ax - cx - b = -d \ (\mathbf{64}, 3). \qquad (2)$$

Adding b to both members of (2),

$$ax - cx = b - d. \qquad (3)$$

This result may be obtained by removing cx from the second member of (1) and writing $-cx$ in the other member, and by removing $-b$ from the first member and writing $+b$ in the other.

RULE.

Remove the term which is to be transposed from the member in which it stands, and write its negative in the other member.

EXERCISES.

Transform each of the following equations into another, the first member of which shall contain all the unknown terms and the second member all the known terms:

1. $2x - 13 = 17 - x$. *Ans.* $2x + x = 17 + 13$.
2. $7x + 6 = 6x + 4$. *Ans.* $7x - 6x = 4 - 6$.
3. $4x - 2 = 9 - 3x$. *Ans.* $4x + 3x = 9 + 2$.
4. $5x + 2a = 3x + 7b$. *Ans.* $5x - 3x = 7b - 2a$.
5. $x - a = (a - b)x$. *Ans.* $x + (b - a)x = a$.
6. $7x - 4 - (3x + 11) = 0$. *Ans.* $7x - 3x = 11 + 4$.

7. $2x - 5(1 - x) - 8 = 0.$

Ans. $2x + 5x = 8 + 5.$

8. $(2x+3)^2 = -8x.$ *Ans.* $4x^2 + 12x + 8x = -9.$

9. $(x^2 - 16)\,9 = 10x^2 - 108.$

Ans. $9x^2 - 10x^2 = 144 - 108.$

10. $\frac{2x}{3} - 1 = \frac{x}{12} - \frac{4}{3}.$ *Ans.* $\frac{2x}{3} - \frac{x}{12} = 1 - \frac{4}{3}.$

11. $\frac{3x^2}{4} - \frac{5y}{6} - 6 = \frac{13}{3} + 3y - \frac{x^2}{2}.$

Ans. $\frac{3x^2}{4} - \frac{5y}{6} - 3y + \frac{x^2}{2} = \frac{13}{3} + 6.$

12. $\frac{x}{2} - \frac{a}{6} - \left(\frac{x}{3} - a\right) + \frac{x}{4} - \frac{a}{20} = 0.$

Ans. $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = \frac{a}{6} + \frac{a}{20} - a.$

160. TH. *If the sign of each term of an equation be changed, the equality will not be destroyed.*

Suppose $x - a = b - y;$

then, by transposition, $y - b = a - x;$

whence, $a - x = y - b$ (**64**, 6).

This theorem may also be proved by multiplying or dividing both members of an equation by -1 (**64**, 3).

CHAPTER VII.

SIMPLE EQUATIONS.

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

161. To solve a simple equation containing only one unknown quantity.

1. Let it be required to solve the equation

$$4x + 10 = 16 + x.$$

By transposition, $4x - x = 16 - 10$;
that is, $3x = 6$;
whence, by division, $x = \frac{6}{3} = 2$ (**64**, 3).

This result may be verified by substituting 2 for x in the given equation. Thus,

$$4 \times 2 + 10 = 16 + 2;$$

that is, $18 = 18$.

2. Let it be required to solve the equation

$$\frac{5}{6}x + 8x - \frac{5}{3} = 103 + \frac{1}{9}x.$$

Clearing of fractions,

$$15x + 144x - 30 = 1854 + 2x;$$

by transposition, $15x + 144x - 2x = 1854 + 30$;
that is, $157x = 1884$;
whence, by division, $x = \frac{1884}{157} = 12$.

RULE.

1. *Clear the equation of fractions, if it has any.*

2. *Transpose every unknown term of the second member to the first, and every known term of the first member to the second; then reduce each member to its simplest form.*

3. *Divide both members by the coefficient of the unknown quantity.*

SCH. Sometimes it is advantageous to combine similar terms of the equation before clearing of fractions, or to clear of fractions *partially*, and then effect some reductions before getting rid of the remaining fractions.

EXERCISES.

1. Solve $$\frac{2x}{3}+\frac{1}{6}-7=\frac{x}{2}.$$

Clearing of fractions,

$$4x+1-42=3x;$$

by transposition, $$4x-3x=42-1;$$

whence, $$x=41.$$

Verification. Putting 41 for x in the given equation,

$$\frac{2\times 41}{3}+\frac{1}{6}-7=\frac{41}{2},\quad\text{or}\quad\frac{41}{2}=\frac{41}{2}.$$

2. Solve $$\frac{(2x-3)\,7}{15}-\frac{1}{10}=4+\frac{3x}{4}.$$

Clearing of fractions,

$$(2x-3)\,28-6=240+45x,$$

that is, $$56x-84-6=240+45x;$$

by transposition, $$56x-45x=240+84+6;$$

that is, $$11x=330;$$

whence, $$x=\frac{330}{11}=30.$$

Verification. Putting 30 for x in the given equation,

$$\frac{(60-3)\,7}{15}-\frac{1}{10}=4+\frac{90}{4},\quad\text{or}\quad\frac{53}{2}=\frac{53}{2}.$$

3. Solve $\dfrac{x+7}{11}-\dfrac{2x-16}{3}+\dfrac{2x+5}{4}=5\tfrac{1}{3}+\dfrac{3x+7}{12}.$

By transposition,

$$\frac{x+7}{11}-\frac{2x-16}{3}+\frac{2x+5}{4}-\frac{3x+7}{12}=5\tfrac{1}{3};$$

Multiplying both members by 12,

$$\frac{12x+84}{11}-8x+64+6x+15-3x-7=64;$$

by transposition and reduction, $\dfrac{12x+84}{11}-5x=-8;$

multiplying by 11, $12x+84-55x=-88;$

by transposition and reduction, $-43x=-172;$

by changing signs, $43x=172;$

whence, $x=\dfrac{172}{43}=4.$

4. Solve $\dfrac{bx}{6a^2}-1=\dfrac{a\,(x-2b)}{4b^2}-\dfrac{a\,(a^2-b^2)}{2b^3}-\dfrac{2}{3}$

By transposition and reduction,

$$\frac{bx}{6a^2}-\frac{ax}{4b^2}=\frac{1}{3}-\frac{2ab}{4b^2}-\frac{a\,(a^2-b^2)}{2b^3};$$

multiplying by $4b^3$,

$$\frac{4b^4x}{6a^2}-abx=\frac{4b^3}{3}-2ab^2-2a^3+2ab^2;$$

that is, $\dfrac{4b^4x}{6a^2}-abx=\dfrac{4b^3}{3}-2a^3;$

multiplying by $6a^2$,

$$4b^4x - 6a^3bx = 8a^2b^3 - 12a^5;$$

whence, $(4b^3 - 6a^3)\,bx = (4b^3 - 6a^3)\,2a^2;$

dividing by $4b^3 - 6a^3$, $bx = 2a^2;$

whence, $x = \frac{2a^2}{b}.$

Solve each of the following equations:

5. $17x - 3 = 45 + 11x.$ *Ans.* $x = 8.$

6. $8x - 5 = 13 - 7x.$ *Ans.* $x = 1\frac{1}{5}.$

7. $2(x - 3) = 3(x - 4).$ *Ans.* $x = 6.$

8. $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 10.$ *Ans.* $x = 24.$

9. $13\frac{3}{4} - \frac{x}{2} = 2x - 8\frac{3}{4}.$ *Ans.* $x = 9.$

10. $5x - 14 = \frac{49x}{3} + 2.$ *Ans.* $x = -1\frac{7}{17}.$

11. $\frac{5}{2}x - \frac{x}{32} - 13 = \frac{5}{8} + \frac{4x}{3}.$ *Ans.* $x = 12.$

12. $6x - 23 = \frac{3}{2}x + 7 + 2x.$ *Ans.* $x = 12.$

13. $2 + \frac{9 - 2x}{3} = 4 - \frac{x + 3}{6}.$ *Ans.* $x = 3.$

14. $2x + \frac{5x}{7} - \frac{x}{3} - \frac{4x}{5} = \frac{1}{4}.$ *Ans.* $x = \frac{105}{664}.$

15. $\frac{10}{3x - 4} = \frac{25}{x + 3}.$ *Ans.* $x = 2.$

16. $3x - 6 - \frac{3x}{4} = \frac{7x}{3} - 12\frac{1}{4} - 5\frac{3}{8}.$ *Ans.* $x = 139\frac{1}{2}.$

17. $\frac{3x}{4} - \frac{5x}{6} - 6 = \frac{13}{3} + 3x - \frac{x}{2}.$ *Ans.* $x = -4.$

18. $\frac{21-5x}{5} - \frac{2x-28}{2} = \frac{8}{5} - x.$ *Ans.* $x = 16\frac{3}{5}.$

19. $\frac{x+3}{4} - 16 = -\frac{x+1}{2} - \frac{x+2}{3}.$ *Ans.* $x = 13.$

20. $\frac{3-x}{2} - \frac{x}{3} = \frac{x-19}{2} - 20.$ *Ans.* $x = 23\frac{1}{4}.$

21. $\frac{7x}{10} - \frac{3x}{5} - \frac{3x}{4} + \frac{7x}{8} = 15.$ *Ans.* $x = 66\frac{2}{3}.$

22. $3x - \frac{x}{2} - 282 = \frac{x}{3} + \frac{x}{4} - 2x.$ *Ans.* $x = 72.$

23. $18 + \frac{36}{125} + \frac{63x}{50} = \frac{33x}{2}.$ *Ans.* $x = 1\frac{1}{5}.$

24. $\left(\frac{2x+5}{2}\right)\left(\frac{2x-3}{2}\right) - (x+5)(x-3) + \frac{3}{4} = 0.$
Ans. $x = 12.$

25. $\frac{x}{2} - \frac{a}{6} - \left(\frac{x}{3} - \frac{a}{12}\right) + \frac{x}{4} - \frac{a}{20} = 0.$ *Ans.* $x = \frac{8a}{25}.$

26. $\frac{3a+x}{x} = \frac{6}{x} + 5.$ *Ans.* $x = \frac{3a-6}{4}.$

27. $3ax + \frac{a}{2} - 3 = bx - a.$ *Ans.* $x = \frac{6-3a}{6a-2b}.$

28. $\frac{x}{a} - \frac{x}{a-b} = \frac{a}{a+b}.$ *Ans.* $x = \frac{a^2(b-a)}{b(a+b)}.$

29. $b - ax = \frac{1}{b} - \frac{x}{a}.$ *Ans.* $x = \frac{a(b^2-1)}{b(a^2-1)}.$

30. $(a+x)(b+x) - \frac{a^2c}{b} = a(b+c) + x^2.$
Ans. $x = \frac{ac}{b}.$

31. $\frac{2x+a}{b} - \frac{x-b}{a} = \frac{3x}{b} + \frac{(a-b)^2}{ab}.$ *Ans.* $x = \frac{2ab}{a+b}.$

32. $\frac{b^2-ax}{b} = b - \frac{a^2-bx}{a}.$ *Ans.* $x = \frac{a^2b}{a^2+b^2}.$

33. $\frac{x}{6} - \frac{x-\frac{1}{2}}{3} - \frac{1}{3}\left(\frac{2}{5} - \frac{x}{3}\right) = 0.$ *Ans.* $x = \frac{3}{5}$.

34. $\frac{x-5\frac{1}{4}}{9} + \frac{15}{16} + \frac{14\frac{1}{4} - \frac{3x}{4}}{4} = 60\frac{1}{4} - \frac{\frac{5x}{4} - 24}{12} - 2\frac{3}{4}x.$

Ans. $x = 21$.

SOLUTION OF PROBLEMS.

162. A **Condition** of a problem is the relation which exists between any two or more of the quantities involved in the problem.

An Explicit Condition is a condition stated in the enunciation of the problem.

An Implicit Condition is a condition deduced from one or more of the explicit conditions.

163. The solution of a problem by Algebra consists of two parts:

1st. *The Algebraic Statement,* which consists in expressing, in algebraic language, the condition or conditions of the problem. This part of the process is sometimes called *putting the problem into an equation* or *into equations.*

2d. *The Solution of the Equation or Equations.*

EXERCISES IN EXPRESSING QUANTITIES.

164. Success in putting problems into equations depends largely upon facility in expressing quantities by means of algebraic symbols. Therefore the student should be drilled in such exercises as the following:

I. If x represents the number of dollars in a certain sum of money, express

(1) Its double, less \$3. *Ans.* $2x - 3$.

(2) Half its value, less $\$\frac{1}{2}$. *Ans.* $\frac{1}{2}x - \frac{1}{2}$.

(3) The number of dimes in one-third of it.

Ans. $\frac{10x}{3}$.

(4) The number of dimes left after paying out 25 cents.

Ans. $10x - 2\frac{1}{2}$.

(5) One man's share, if three times the amount, increased by $5, is divided equally among 9 men.

Ans. $\frac{3x + 5}{9}$.

(6) A boy's share, if the amount, increased by 50 cents, is divided among 3 boys.

Ans. $\frac{x + \frac{1}{2}}{3}$.

(7) The price of one loaf, if the amount buys 10 loaves.

Ans. $\frac{x}{10}$.

(8) The charge per mile, if the amount buys a railway ticket for 33 miles.

Ans. $\frac{x}{33}$.

(9) The charge for 11 miles, at the same rate.

Ans. $\frac{x}{3}$.

II. If x represents the number of years in a man's age, express

(1) His age 5 years ago. *Ans.* $x - 5$.

(2) His age 5 years hence. *Ans.* $x + 5$.

(3) The present age of his wife, whose age was two-thirds of his ten years ago.

Ans. $\frac{2x + 10}{3}$.

(4) The age of his wife 10 years hence. *Ans.* $\frac{2x + 40}{3}$.

III. If x represents the number of yards in the length of a line, express

(1) Its half, diminished by an inch. *Ans.* $\frac{x}{2} - \frac{1}{36}$.

(2) 111 feet, diminished by three times the length.
Ans. $37 - 3x$ yds.

(3) The result obtained by taking from the length one-half of it, one inch, and one-fifth of the remainder.
Ans. $\frac{2x}{5} - \frac{1}{45}$ yds.

(4) The time required to traverse it, at the rate of a yards per minute.
Ans. $\frac{x}{a}$ minutes.

(5) The time required to traverse it by a man walking at the rate of a miles in c hours.
Ans. $\frac{cx}{1760a}$ hours.

IV. If a train moves at the rate of x miles an hour,

(1) How far does it go in 20 minutes? *Ans.* $\frac{x}{3}$ miles.

(2) How far does it go in a hours? *Ans.* ax miles.

(3) How far does it go in the time that a man, who walks a miles per hour, takes to walk a mile?
Ans. $\frac{x}{a}$ miles.

(4) How far does it go in the time the same man takes to walk b miles?
Ans. $\frac{bx}{a}$ miles.

(5) How long does it take to go a mile?
Ans. $\frac{1}{x}$ of an hour.

(6) How long does it take to go b miles?
Ans. $\frac{b}{x}$ hours.

(7) How long does it take to go the distance which a man, who walks a miles an hour, goes in 15 hours and 20 minutes?
Ans. $\frac{46a}{3x}$ hours.

(8) What is its rate per minute? *Ans.* $\frac{x}{60}$ of a mile.

(9) What is its rate per minute in feet? *Ans.* $88x$ feet.

(10) If the train is a yards long, how many minutes does it take to pass a given post? *Ans.* $\frac{3a}{88x}$.

(11) How many minutes does it take to pass another train, b yards long, at rest? *Ans.* $\frac{3(a+b)}{88x}$.

V. (1) Express x per cent. of a. *Ans.* $\frac{ax}{100}$.

(2) If x per cent. of b is c, what is x per cent. of a?

Ans. $\frac{ac}{b}$.

(3) What must a merchant charge for an article which cost him $\$x$, in order to make 5 per cent. profit on it?

Ans. $\$\frac{21x}{20}$.

(4) If x gallons of wine be taken out of a full cask that holds a gallons, and the cask be then filled with water, how many gallons of wine will there be in b gallons of the mixture? *Ans.* $\left(\frac{a-x}{a}\right)b$.

EXERCISES IN SOLVING PROBLEMS.

165. 1. If 13 be added to the double of a certain number, the sum will be 57. What is that number?

Let x denote the required number; then $2x$ will be the double of that number, and $2x + 13$ will be the sum obtained by adding 13 to the double of the number. By the condition of the problem, this sum is equal to 57;

$$\therefore \quad 2x + 13 = 57;$$

whence,

$$x = 22.$$

2. If one-third of a certain number be added to one-half of it, the sum will be 30. What is that number?

Let x denote the required number; then $\frac{x}{2} + \frac{x}{3}$ will be the sum of one-half of that number and one-third of it. By the problem, this sum is equal to 30;

$$\therefore \quad \frac{x}{2} + \frac{x}{3} = 30;$$

whence, $x = 36.$

3. The sum of one-half of a number, one-third of it, and 5 is equal to twice the number, diminished by 30. What is that number?

Let x denote the required number; then $\frac{x}{2} + \frac{x}{3} + 5$ will be the sum of one-half of that number, one-third of it, and 5; and $2x - 30$ will be twice the number, diminished by 30. By the problem, these expressions are equal;

$$\therefore \quad \frac{x}{2} + \frac{x}{3} + 5 = 2x - 30;$$

whence, $x = 30.$

4. A took from a purse \$15 and one-sixth of the remainder; then B took \$12 and one-fourth of the remainder; they then found that they had taken equal amounts. How many dollars were in the purse at first?

Let $x =$ the number of dollars in the purse at first; then $15 + \frac{1}{6}(x - 15) =$ the number of dollars A took; and the number of dollars remaining is $x - [15 + \frac{1}{6}(x - 15)] = \frac{5}{6}(x - 15)$; hence B took $12 + \frac{1}{4}[\frac{5}{6}(x - 15) - 12]$.

$$\therefore \quad 15 + \tfrac{1}{6}(x - 15) = 12 + \tfrac{1}{4}[\tfrac{5}{6}(x - 15) - 12];$$

whence, $15 + \frac{1}{6}(x - 15) = 12 + \frac{5}{24}(x - 15) - 3;$

whence, $x = 159.$

5. Three persons, A, B, and C, owned a ship's cargo. A owned one-fourth, B, one-fifth, and C's share was worth \$1100. What were A's and B's shares worth?

Let $x =$ the value of the whole cargo;

then $\frac{x}{4} =$ the value of A's share,

and $\frac{x}{5} =$ the value of B's share;

$$\therefore \quad \frac{x}{4} + \frac{x}{5} + 1100 = x \ (\mathbf{64}, 1);$$

whence, $x = 2000$;

hence, A's share is worth \$500, and B's, \$400.

6. Divide the number 75 into two parts, one of which shall be 4 times the other.

Let $x =$ the smaller part;

then $4x =$ the greater;

$$\therefore \quad x + 4x = 75;$$

whence, $x = 15, \quad \text{and} \quad 4x = 60.$

7. Divide the number a into two parts, one of which shall be m times the other.

Let $x =$ the smaller part;

then $mx =$ the greater;

$$\therefore \quad x + mx = a;$$

whence, $x = \frac{a}{1+m}, \quad \text{and} \quad mx = \frac{am}{1+m}.$

As a and m are general symbols, the solution of this problem furnishes formulas which may be used in solving any similar problems. For example, to obtain the answer to Prob. 6, substitute 75 for a, and 4 for m. Thus,

$$x = \frac{75}{1+4} = 15, \quad \text{and} \quad 4x = \frac{75 \times 4}{1+4} = 60.$$

A problem involving particular numbers is said to be *generalized* when general symbols are used instead of those numbers. Thus, Prob. 6 is generalized by putting a instead of 75, and m instead of 4.

8. A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1 lb. per head; after being at sea 20 days, she encounters a storm, in which 5 of her crew are lost, and damage sustained which will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ lb. Find the original number of the crew.

Let $x =$ the number of men in the crew at first;
then $60x =$ the number of lbs. of biscuit on sailing,
and $40x =$ " " " " 20 days later.

After the storm, the number of men in the crew was $x - 5$, and they had to remain at sea 64 days, at a daily allowance of $\frac{5}{7}$ lb. per head.

In that time they used $\frac{5}{7}(x-5)\,64$ lbs. of biscuit; then, since they used all that remained at the end of 20 days, it follows that

$$40x = \tfrac{5}{7}(x-5)\,64;$$

whence,
$$x = 40.$$

9. A ship sails with a supply of biscuit for a days, at a daily allowance of m lbs. per head; after being at sea b days, she encounters a storm, in which n of her crew are lost, and damage sustained which will cause a delay of c days, and it is found that each man's allowance must be reduced to p lbs. Find the original number of the crew.

Let $x =$ the number of men in the crew at first;
then $max =$ the number of lbs. of biscuit on sailing,
and $m(a-b)\,x =$ " " " " b days later.

After the storm, the number of men in the crew was

$x - n$, and they had to remain at sea $a - b + c$ days, at a daily allowance of p lbs. per head.

In that time they used $p(x-n)(a-b+c)$ lbs. of biscuit; then, since they used all that remained at the end of b days, it follows that

$$m(a-b)x = p(x-n)(a-b+c);$$

whence, $$x = \frac{np(a-b+c)}{(p-m)(a-b)+cp}.$$

If $a = 60$, $m = 1$, $b = 20$, $n = 5$, $c = 24$, and $p = \frac{5}{7}$;

then $$x = \frac{5 \times \frac{5}{7}(60-20+24)}{(\frac{5}{7}-1)(60-20)+24\times\frac{5}{7}} = \frac{\frac{25}{7}\times 64}{(-\frac{2}{7})40+\frac{120}{7}}$$

$$= \frac{25\times 64}{-80+120} = \frac{1600}{40} = 40.$$

10. If I add \$45 to the money in my purse, it will then contain four times as much as it now does. What is the sum in my purse? *Ans.* \$15.

11. If one-fourth of a certain number be added to one-seventh of it, the sum will be 176. What is that number? *Ans.* 448.

12. A and B together gave \$50 for charitable purposes, A contributing \$12 more than B. What did each give? *Ans.* A, \$31; and B, \$19.

13. One-sixth of a certain number exceeds one-tenth of it by 28. What is that number? *Ans.* 420.

14. A man's age is three times that of his son, and the sum of their ages is 48 years. What is the age of each? *Ans.* Man's, 36 years; son's, 12 years.

15. A, B, and C together gave \$150 for charitable purposes, A contributing \$10 more than B, and C, \$25 more than A. What did each give? *Ans.* A, \$45; B, \$35; C, \$70.

16. Divide 40 into two parts, so that one part shall be three-fifths of the other. *Ans.* 15, 25.

17. Divide 60 into two parts, so that one-third of one part shall be equal to one-half of the other. *Ans.* 36, 24.

18. Divide 192 into two parts, so that one-half of the greater shall exceed one-third of the less by 21.
Ans. 102, 90.

19. If 36 be subtracted from a certain number, the remainder will be two-fifths of that number. What is the number? *Ans.* 60.

20. A man's age is 42 years, and the age of his son is 6 years; in how many years will the father be three times as old as his son? *Ans.* 12.

21. A's age is 72 years, and B's age is 48 years. How many years since A was five times as old as B? *Ans.* 42.

22. A man is four times as old as his son; but twelve years ago he was sixteen times as old. What are their ages?
Ans. 60 years, and 15 years.

23. Find a number, such that one half of it exceeds 100 as much as one-third of it is exceeded by 100. *Ans.* 240.

24. The difference between two numbers is 20, and one-seventh of the greater exceeds one-tenth of the less by 5. What are the numbers? *Ans.* 70 and 50.

25. The difference between two numbers is 10, and half of one of them is greater than twice the other by 2. What are the numbers? *Ans.* 12 and 2.

26. Divide 70 into two parts, such that the sum of the quotients obtained by dividing one part by 2 and the other by 3 shall be 31. *Ans.* 46 and 24.

27. A's annual income is $450 less than B's; A lays by half of his, and thereby saves in two years $200 more than B, who lays by only one-sixth of his income. What was the income of each? *Ans.* A's, $525; B's, $975.

28. A man sold 180 acres more than one-sixth of his estate, and had remaining 20 acres less than one-third of it. Of how many acres did the estate consist? *Ans.* 320.

29. A man bought a case of oranges at 24 cts. a dozen; he found 50 spoiled, and selling the remainder at the rate of 3 for 8 cts., made $5.16 profit: how many oranges were in the case? *Ans.* 974.

30. A was sent on an errand to a neighboring village; when he had been gone 8 minutes, B ran after him with a forgotten parcel, and, by going twice as fast as A, overtook him a mile from home: at what rate did A travel?
Ans. $3\frac{3}{4}$ miles per hour.

31. If A can perform a piece of work in 9 days, and B in 12 days, in what time will they perform it, if they work together? *Ans.* $5\frac{1}{7}$ days.

32. A colonel on attempting to arrange his men in the form of a solid square found that he had 31 men over, and that he could add one more man to each side of the square, if he had 24 men more in his regiment: how many men were there in the regiment? *Ans.* 760.

33. A starts from a certain place, and travels at the rate of 7 miles in 5 hours; B starts from the same place 8 hours after A, and travels in the same direction at the rate of 5 miles in 3 hours: how far will A travel before he is overtaken by B? *Ans.* 70 miles.

34. If 20 men, 40 women, and 50 boys receive $250 among them for a week's work, and 2 men receive as much as 3 women or 5 boys, what does each woman receive for a week's work? *Ans.* $2.50.

35. Divide 100 into two such parts that the difference of their squares shall be 1000. *Ans.* 55 and 45.

36. Two persons who are 154 miles apart start at the same time with a design to meet; one travels at the rate of 3 miles in 2 hours, and the other at the rate of 5 miles in 4 hours: when will they meet? *Ans.* 56 hours after starting.

37. A, B, and C together can perform a piece of work in 24 days; A can do half as much work as B, and B can do

half as much as C: in what time could each alone perform the work?

Ans. A, in 168 days; B, in 84 days; C, in 42 days.

38. A market woman bought a certain number of eggs at the rate of 5 for two cents; she sold half of them at the rate of 2 for a cent, and half at the rate of 3 for a cent, and gained 4 cts: how many eggs did she buy? *Ans.* 240.

39. A man divided his estate among his four children, A, B, C, and D; to A he gave $800 less than half of the estate, to B $120 more than one-fourth of it, to C half as much as to A, and to D two-thirds as much as to B: how much did he give to each?

Ans. To A $2200, to B $1620, to C $1100, to D $1080.

40. A man walking at the rate of 4 miles an hour observes that a train 88 yards long passes him in 5 seconds: at what rate is it moving? *Ans.* 40 miles an hour.

41. An estate of 864 acres was divided among A, B, and C, so that A's share was $\frac{5}{11}$ of B's, and C's share was equal to the sum of the others. Find the shares.

Ans. 135 A., 297 A., 432 A.

42. In a half-mile walking match between A and B, A won by 155 yards. B's step was 5 inches longer than A's, but A made 16 steps while B made 11. Find A's step.

Ans. $25\frac{5}{24}$ inches.

43. A train 66 yards long, which had come 50 miles from the terminus in one hour, met another 110 yards long, which it passed in 5 seconds; at what rate was the latter moving, and where will it meet a freight train, moving at the rate of 20 miles an hour, which left the terminus half-an-hour after the first?

Ans. 22 miles per hour; $29\frac{1}{21}$ miles from terminus.

44. A person walked to the top of a mountain at the rate of $2\frac{1}{3}$ miles an hour, and down the same way at $3\frac{1}{2}$ miles an

hour, and was out 5 hours. Find the distance to the summit. *Ans.* 7 miles.

45. A farmer calculated that he had sufficient food to support p cows through the winter. After feeding them 9 days, the cattle plague broke out and carried off r of them daily for 4 days. In consequence also of a long continuance of frost he was obliged to feed the remaining stock three days longer than he had expected, when the food was exhausted. How many days would the food have lasted had no deaths occurred? *Ans.* $\frac{3p + 30r}{4r}$.

46. If m women can do the work of n boys, and if p men can do the work of q women, what is a woman's share of $500, which is paid to c boys, b women, and a men for doing a piece of work? *Ans.* $\frac{500np}{naq + nbp + mcp}$ dollars.

SIMPLE SIMULTANEOUS EQUATIONS.

166. The equation

$$ax + by = c,$$

in which a, b, and c may be either positive or negative, integral or fractional, may be considered as *the standard form* for simple equations containing only two unknown quantities, for all such equations may be reduced to that form; the equation

$$ax + by + cz = d$$

may be considered as *the standard form* for simple equations containing only three unknown quantities; and so on.

Thus, by clearing of fractions, transposing, and uniting similar terms, the equation

$$\frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4}$$

can be reduced to $32x - 15y = 74,$

and the equation $\frac{x+z-2y}{3} = \frac{z-y-1}{2}$

can be reduced to $2x - y - z = -3.$

167. Simultaneous Equations are equations which are to be satisfied by the *same values* of the unknown quantities.

168. Suppose we have an equation containing two unknown quantities; for example,

$$2x + 3y = 13. \qquad (1)$$

For every value which may be assigned to one of the unknown quantities we can determine the corresponding value of the other, and thus find as many pairs of values as we please. Thus,

if $y = 2$, then $x = 3\frac{1}{2}$;
if $y = 3$, then $x = 2$;
if $y = 4$, then $x = \frac{1}{2}$; and so on.

But suppose we have a second equation, as

$$3x + 2y = 12, \qquad (2)$$

which is also to be satisfied.

Although the equation, considered by itself, may be satisfied by an infinite number of pairs of values of x and y, yet there is only *one pair* of these values that will satisfy both equations *simultaneously;* for multiplying (1) by 2 and (2) by 3, we have

$$4x + 6y = 26, \qquad (3)$$

$$9x + 6y = 36. \qquad (4)$$

Subtracting (3) from (4), $5x = 10$ (**64**, 3);

whence, $x = 2.$

The value of y may now be found by substituting 2 for x in either of the given equations. Substituting 2 for x in (1),

$$4 + 3y = 13;$$

whence,
$$y = 3.$$

Hence, if *both* equations are to be satisfied at the same time, x must be equal to 2, and y must be equal to 3.

169. It is sometimes impossible to satisfy two or more equations involving two or more unknown quantities by the same values of the unknown quantities. Thus,

$$2x + 3y = 7, \tag{1}$$

and
$$4x + 6y = 13 \tag{2}$$

cannot be satisfied by the same values of x and y, for the first member of (2) is double the first member of (1), if x and y have the same values in the two equations; but the second member of (2) is *not* double the second member of (1).

Such equations are said to be *Contradictory*, or *Incompatible.*

170. Two equations are said to be **Independent**, if neither of them can be deduced from the other. Thus,

$$x + 3y = 19 \quad \text{and} \quad 2x + 5y = 33$$

are independent.

Independent equations express conditions *essentially different.*

171. Two equations are said to be **Dependent**, if either of them can be deduced from the other. Thus,

$$x + 3y = 19 \quad \text{and} \quad 3x + 9y = 57$$

are dependent, for the second may be deduced from the first by multiplying both of its members by 3,

Dependent equations express the same conditions under different forms.

ELIMINATION.

172. To Eliminate an unknown quantity from two equations is to combine the two equations in such a manner as to obtain a third equation which shall not contain that unknown quantity.

ELIMINATION BY ADDITION OR SUBTRACTION.

173. 1. Let it be required to eliminate y from the equations

$$5x + 4y = 23, \qquad (1)$$

$$2x - 3y = 12. \qquad (2)$$

Multiplying (1) by 3, and (2) by 4,

$$15x + 12y = 69, \qquad (3)$$

$$8x - 12y = 48. \qquad (4)$$

Adding (3) and (4), $23x = 117.$

In this example, y has been eliminated by *addition*.

2. Let it be required to eliminate y from the equations

$$6x + 21y = 87, \qquad (1)$$

$$3x + 5y = 27. \qquad (2)$$

Dividing (1) by 3, $2x + 7y = 29.$ (3)

Multiplying (3) by 5, and (2) by 7,

$$10x + 35y = 145, \qquad (4)$$

$$21x + 35y = 189. \qquad (5)$$

Subtracting (4) from (5), $11x = 44.$

In this example, y has been eliminated by *subtraction*.

3. Let it be required to eliminate y from the equations

$$\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}, \tag{1}$$

$$\frac{x+13}{4}+\frac{2x+y}{8}=\frac{4+2y}{3}. \tag{2}$$

Reducing these equations to the standard form **(166)**, we obtain

$$32x-15y=74 \tag{3}$$

$$12x-13y=-46. \tag{4}$$

Multiplying (3) by 13, and (4) by 15,

$$416x-195y=962,$$

$$180x-195y=-690.$$

Subtracting (6) from (5), $236x=1652.$

RULE.

1. *Reduce the given equations to the standard form* **(166)**.

2. *Prepare them, by multiplication or division, so that the coefficients of the quantity to be eliminated shall be numerically equal.*

3. *If these coefficients have like signs, subtract one of the equations from the other; but if they have unlike signs, add the equations.*

SCH. 1. In preparing the equations by multiplication, it is best to divide the l. c. m. of the coefficients of the quantity to be eliminated by each of these coefficients; the quotients thus obtained will be the *least multipliers* that can be used.

SCH. 2. It is generally convenient to clear the equations of fractions, if they have any. This is not necessary, however; for, if the quantity to be eliminated has fractional coefficients in the two equations, they may be reduced to equivalent fractions having a common denominator. These equations may then be prepared for elimination by multiplication or division, according to the second step in the rule. Thus, if it be required to eliminate y from the equations

$$\tfrac{3}{5}x + \tfrac{7}{2}y = 16, \tag{1}$$

$$\tfrac{3}{2}x - \tfrac{2}{3}y = 20, \tag{2}$$

we may proceed as follows:

$$\tfrac{7}{2} = \tfrac{21}{6}, \quad \text{and} \quad \tfrac{2}{3} = \tfrac{4}{6};$$

hence, (1) and (2) may be expressed thus,

$$\tfrac{3}{5}x + \tfrac{21}{6}y = 16, \tag{3}$$

$$\tfrac{3}{2}x - \tfrac{4}{6}y = 20. \tag{4}$$

Multiplying (3) by 4, and (4) by 21,

$$\tfrac{12}{5}x + 14y = 64, \tag{5}$$

$$\tfrac{63}{2}x - 14y = 420. \tag{6}$$

Adding (5) and (6), $\tfrac{339}{10}x = 484.$

ELIMINATION BY SUBSTITUTION.

174. Let it be required to eliminate y from the equations

$$5x + 4y = 23, \tag{1}$$

$$2x + 3y = 12. \tag{2}$$

From (1), $y = \dfrac{23 - 5x}{4}.$

Substituting $\dfrac{23 - 5x}{4}$ for y in (2),

$$2x + \frac{3\,(23 - 5x)}{4} = 12.$$

In this example, y has been eliminated by *substitution.*

RULE.

From one of the given equations find the value of the quantity to be eliminated, in terms of the other quantities in that equation; then substitute this value for the same quantity in the other equation.

ELIMINATION BY COMPARISON.

175. Let it be required to eliminate y from the equations

$$12x - 13y = 10, \tag{1}$$

$$9x + 28y = 83. \tag{2}$$

From (1),
$$y = \frac{12x - 10}{13};$$

from (2),
$$y = \frac{83 - 9x}{28};$$

$$\therefore \quad \frac{12x - 10}{13} = \frac{83 - 9x}{28} \quad \textbf{(64}, 2).$$

In this example, y has been eliminated by *comparison.*

RULE.

From each of the given equations find the value of the quantity to be eliminated, in terms of the other quantities; then equate the values thus found.

SCH. It is sometimes convenient to eliminate by finding, from each equation, the value of some *expression* containing the quantity to be eliminated, in terms of the other quantities, and equating the values thus found. For example, let it be required to eliminate y from the equations $5x + 2 + 3y = 25$ and $8x - (2 + 3y) = 18$.

From the first equation, $2 + 3y = 25 - 5x$, and from the second, $2 + 3y = 8x - 18$;

$$\therefore \quad 25 - 5x = 8x - 18.$$

PAIRS OF SIMPLE EQUATIONS

CONTAINING ONLY TWO UNKNOWN QUANTITIES.

176. To solve a pair of simple equations containing only two unknown quantities.

1. Let it be required to solve the equations

$$5x + 4y = 23, \quad (1)$$

$$2x + 3y = 12. \quad (2)$$

Multiplying (1) by 3, and (2) by 4,

$$15x + 12y = 69, \quad (3)$$

$$8x + 12y = 48. \quad (4)$$

Subtracting (4) from (3), $7x = 21$;

whence, $x = 3$.

Hence, by (2), $6 + 3y = 12$;

whence, $y = 2$.

2. Let it be required to solve the equations

$$\frac{x+y}{2} - \frac{x-y}{3} = 8, \quad (1)$$

$$\frac{x+y}{3} - \frac{x-y}{4} = 5. \quad (2)$$

Clearing these equations of fractions and uniting similar terms,

$$x + 5y = 48, \quad (3)$$

$$x + 7y = 60. \quad (4)$$

Subtracting (3) from (4), $2y = 12$;

whence, $y = 6$.

Hence, by (4), $x + 42 = 60$;

whence, $x = 18$.

3. Let it be required to solve the equations

$$12x - 13y = 10, \qquad (1)$$

$$9x + 28y = 83. \qquad (2)$$

From (1), $$y = \frac{12x - 10}{13}. \qquad (3)$$

Substituting $\frac{12x - 10}{13}$ for y in (2),

$$9x + \frac{28\,(12x - 10)}{13} = 83;$$

whence, $$117x + 336x - 280 = 1079;$$

whence, $$453x = 1359;$$

whence, $$x = 3.$$

Hence, by (3), $$y = \frac{36 - 10}{13} = 2.$$

4. Let it be required to solve the equations

$$\frac{x}{5} + \frac{y}{6} = 18, \qquad (1)$$

$$\frac{x}{2} - \frac{y}{4} = 21. \qquad (2)$$

From (1), $$y = 108 - \frac{6x}{5}; \qquad (3)$$

from (2), $$y = 2x - 84; \qquad (4)$$

$$\therefore\ 2x - 84 = 108 - \frac{6x}{5};$$

whence, $$x = 60.$$

Hence, by (4), $$y = 120 - 84 = 36.$$

5. Let it be required to solve the equations

$$\frac{12}{x} + \frac{8}{y} = 8, \qquad (1)$$

$$\frac{27}{x} - \frac{12}{y} = 3. \qquad (2)$$

Multiplying (1) by 3, and (2) by 2,

$$\frac{36}{x}+\frac{24}{y}=24, \qquad (3)$$

$$\frac{54}{x}-\frac{24}{y}=6. \qquad (4)$$

Adding (3) to (4), $$\frac{90}{x}=30;$$

whence, $$\frac{1}{x}=\frac{1}{3}, \quad \text{or} \quad x=3.$$

Substituting $\frac{1}{3}$ for $\frac{1}{x}$ in (1),

$$4+\frac{8}{y}=8;$$

whence, $$\frac{1}{y}=\frac{1}{2}, \quad \text{or} \quad y=2.$$

If x and y are regarded as the unknown quantities, (1) and (2) do not properly belong to the present chapter; for clearing them of fractions, we obtain

$$12y+8x=8xy,$$
$$27y-12x=3xy,$$

which are equations of the second degree. But if $\frac{1}{x}$ and $\frac{1}{y}$ are regarded as the unknown quantities, the given equations are of the first degree. When such equations as these occur in the present chapter, let it be understood that $\frac{1}{x}$ and $\frac{1}{y}$ are to be regarded as the unknown quantities.

RULE.

1. *Combine the given equations so as to eliminate one of the unknown quantities.*

2. *Find the value of the unknown quantity in the resulting equation.*

3. *Substitute this value for this unknown quantity in any one of the equations containing both of the unknown quantities, and solve the resulting equation.*

EXERCISES.

Solve the following pairs of simultaneous equations:

1. $\left\{\begin{array}{r} 3x+4y=26 \\ x+3y=17 \end{array}\right\}.$ *Ans.* $x=2,\ y=5.$

2. $\left\{\begin{array}{r} x+2y=19 \\ 4x+5y=55 \end{array}\right\}.$ *Ans.* $x=5,\ y=7.$

3. $\left\{\begin{array}{r} 2x-5y=9 \\ x+y=15 \end{array}\right\}.$ *Ans.* $x=12,\ y=3.$

4. $\left\{\begin{array}{r} x+9y=21 \\ 3x+5y=19 \end{array}\right\}.$ *Ans.* $x=3,\ y=2.$

5. $\left\{\begin{array}{r} 3x+2y=46 \\ 4x+y=48 \end{array}\right\}.$ *Ans.* $x=10,\ y=8.$

6. $\left\{\begin{array}{r} 2x-y=16 \\ 3x-7y=35 \end{array}\right\}.$ *Ans.* $x=7,\ y=-2.$

7. $\left\{\begin{array}{r} 3x+5y=37 \\ 2x+3y=23 \end{array}\right\}.$ *Ans.* $x=4,\ y=5.$

8. $\left\{\begin{array}{r} 5x+7y=0 \\ 12x-y=89 \end{array}\right\}.$ *Ans.* $x=7,\ y=-5.$

9. $\left\{\begin{array}{r} 5x-4y=28 \\ 8x+3y=-21 \end{array}\right\}.$ *Ans.* $x=0,\ y=-7.$

10. $\left\{\begin{array}{l} 25-5x=4y \\ 3x-7y-15=0 \end{array}\right\}.$ *Ans.* $x=5,\ y=0.$

11. $\left\{\begin{array}{l} 7y+2x=80-4x \\ 4x+2y=66-9y \end{array}\right\}.$ *Ans.* $x=11,\ y=2.$

12. $\left\{\begin{array}{l} 13x+10y-103=0 \\ 9x-y=0 \end{array}\right\}.$ *Ans.* $x=1,\ y=9.$

13. $\left\{\begin{array}{r} 3y-x=14 \\ 2y+3x=13 \end{array}\right\}.$ *Ans.* $x=1,\ y=5.$

14. $\left\{\begin{array}{l} \dfrac{7x+3y}{4}-\dfrac{x+y}{8}=5 \\ 56-3x=7y \end{array}\right\}.$ *Ans.* $x=0,\ y=8.$

15. $$\left\{\begin{array}{l} \frac{x}{2} + \frac{y}{4} = 7 \\ \frac{y}{8} + \frac{x}{3} = 4 \end{array}\right\}.$$ *Ans.* $x = 6$, $y = 16$.

16. $$\left\{\begin{array}{l} \frac{y}{5} - \frac{x}{10} = 1 \\ \frac{2y}{3} - \frac{x}{4} = 5 \end{array}\right\}.$$ *Ans.* $x = 20$, $y = 15$.

17. $$\left\{\begin{array}{l} \frac{2y + x}{9} - \frac{x}{4} = 1 \\ \frac{y - x}{3} + \frac{y}{7} = 2 \end{array}\right\}.$$ *Ans.* $x = 4$, $y = 7$.

18. $$\left\{\begin{array}{l} \frac{x}{5} - \frac{x - 2y}{7} = 0 \\ \frac{3x + 5y}{10} = 1 + \frac{x + y}{8} \end{array}\right\}.$$ *Ans.* $x = 10$, $y = -2$.

19. $$\left\{\begin{array}{l} \frac{x - 2y}{2} - \frac{1}{9}(2x - 7) = -9 \\ y = x + 17 \end{array}\right\}.$$ *Ans.* $x = -10$, $y = 7$.

20. $$\left\{\begin{array}{l} \frac{x}{4} - \frac{y}{12} - \frac{7x - 3y}{15} = \frac{1}{10} \\ \frac{x}{4} - \frac{y}{10} = \frac{3}{10} \end{array}\right\}.$$ *Ans.* $x = 6$, $y = 12$.

21. $$\left\{\begin{array}{l} \frac{33x - y}{8} = 21 \\ 2y + x - \frac{5y}{3} = 4 \end{array}\right\}.$$ *Ans.* $x = 5$, $y = -3$.

22. $$\left\{\begin{array}{l} \frac{2x - y}{5} + \frac{x}{4} = 2\frac{1}{2} \\ \frac{x - 2y}{8} + \frac{y}{7} = 0 \end{array}\right\}.$$ *Ans.* $x = 6$, $y = 7$.

23. $$\left\{\begin{array}{l}\frac{3y-5x}{2}+3=\frac{x+2y}{5}\\ 8-\frac{y-2x}{4}=\frac{x}{3}+\frac{y}{2}\end{array}\right\}.$$ *Ans.* $x=6,\ y=12.$

24. $$\left\{\begin{array}{l}\frac{4}{7}\left(\frac{2x}{3}-\frac{5y}{12}\right)-\frac{2}{23}\left(\frac{3x}{2}-\frac{y}{3}\right)=2\\ 1-\frac{2y}{x+y}=\frac{1}{5}\end{array}\right\}.$$

Ans. $x=18,\ y=12.$

25. $$\left\{\begin{array}{l}11x+13y=4a\\ 12y-\ \ 6x=\ \ a\end{array}\right\}.$$ *Ans.* $x=y=\frac{a}{6}.$

26. $$\left\{\begin{array}{l}\frac{y}{m}+\frac{x}{n}=2\\ \frac{y}{m}-\frac{x}{n}=1\end{array}\right\}.$$ *Ans.* $x=\frac{n}{2},\ y=\frac{3m}{2}.$

27. $$\left\{\begin{array}{l}\frac{x}{m}-\frac{y}{n}=1\\ \frac{x}{3m}-\frac{y}{6n}=\frac{2}{3}\end{array}\right\}.$$ *Ans.* $x=3m,\ y=2n.$

28. $$\left\{\begin{array}{l}ax-by=c\\ mx+ny=d\end{array}\right\}.$$

Ans. $x=\frac{cn+bd}{bm+an},\ y=\frac{ad-cm}{bm+an}.$

29. $$\left\{\begin{array}{l}\frac{x}{a+c}+\frac{y}{b+c}=2\\ \frac{bx-ay}{(a-b)c}=-1\end{array}\right\}.$$

Ans. $x=a+c,\ y=b+c.$

30. $$\left\{\begin{array}{l}\frac{x}{a-b}+\frac{y}{a+b}=2a\\ y-x=4ab\end{array}\right\}.$$

Ans. $x=(a-b)^2,\ y=(a+b)^2.$

31. $\left\{\begin{array}{l} \dfrac{x-1}{y} = \dfrac{1}{7} \\ \dfrac{x}{y+1} = \dfrac{1}{6} \end{array}\right\}.$ *Ans.* $x = 6,\ y = 35.$

32. $\left\{\begin{array}{l} \dfrac{3x-2y}{4} + \dfrac{5x+2y}{6} = 5\frac{1}{2} \\ \dfrac{x+2y}{7} - \dfrac{3x-2y}{2} = 1 \end{array}\right\}.$ *Ans.* $x = 4,\ y = 5.$

33. $\left\{\begin{array}{l} \dfrac{4}{x-2} + \dfrac{6}{y} = 5\frac{1}{2} \\ \dfrac{4}{y} - \dfrac{1}{x-2} = 0 \end{array}\right\}.$ *Ans.* $x = 3,\ y = 4.$

34. $\left\{\begin{array}{l} \dfrac{9}{x} - \dfrac{4}{y} = 1 \\ \dfrac{18}{x} + \dfrac{20}{y} = 16 \end{array}\right\}.$ *Ans.* $x = 3,\ y = 2.$

35. $\left\{\begin{array}{l} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{8}{15} \\ \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{2}{15} \end{array}\right\}.$ *Ans.* $x = 3,\ y = 5.$

36. $\left\{\begin{array}{l} \dfrac{7}{x} - \dfrac{5}{y} = \dfrac{1}{6} \\ \dfrac{3}{x} + \dfrac{4}{y} = 1\frac{2}{21} \end{array}\right\}.$ *Ans.* $x = 7,\ y = 6.$

37. $\left\{\begin{array}{l} \dfrac{m}{x} + \dfrac{n}{y} = b \\ \dfrac{m}{y} + \dfrac{n}{x} = a \end{array}\right\}.$ *Ans.* $x = \dfrac{m^2-n^2}{bm-an},\ y = \dfrac{m^2-n^2}{am-bn}.$

38. $\left\{\begin{array}{l} \dfrac{m}{x} - \dfrac{n}{y} = a \\ \dfrac{x}{y} = \dfrac{q}{p} \end{array}\right\}.$ *Ans.* $x = \dfrac{mp-nq}{ap},\ y = \dfrac{mp-nq}{aq}.$

GROUPS OF SIMPLE EQUATIONS

CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

177. To solve a group of simple equations containing more than two unknown quantities.

1. Let it be required to solve the equations

$$8x - 19y + 16z = 2, \quad (1)$$
$$5x - 8y + 4z = 3, \quad (2)$$
$$11x - 7y + 7z = 26. \quad (3)$$

Combining (1) and (2), also (2) and (3), eliminating z in each case, we have the new group

$$12x - 13y = 10, \quad (4)$$
$$9x + 28y = 83. \quad (5)$$

Combining (4) and (5), eliminating x,

$$151y = 302;$$

whence, $y = 2.$

Substituting 2 for y in (5),

$$9x + 56 = 83;$$

whence, $x = 3.$

Substituting 3 for x and 2 for y in (2),

$$15 - 16 + 4z = 3;$$

whence, $z = 1.$

2. Let it be required to solve the equations

$$4x - 7y + 2z - 6t = -24, \quad (1)$$
$$4x - 12y + 14z + 6t = 26, \quad (2)$$
$$3x - 5y + 2z - 2t = -7, \quad (3)$$
$$5x + 3y + 3z + 4t = 40. \quad (4)$$

Combining (1) and (2), (2) and (3), and (3) and (4), eliminating t by each combination, we have the new group

$$8x - 19y + 16z = 2, \tag{5}$$
$$13x - 27y + 20z = 5, \tag{6}$$
$$11x - 7y + 7z = 26. \tag{7}$$

Combining (5) and (6), also (6) and (7), eliminating z by each combination, we have the new group

$$12x - 13y = 10, \tag{8}$$
$$129x + 49y = 485. \tag{9}$$

Solving (8) and (9), we find

$$x = 3, \quad \text{and} \quad y = 2.$$

Substituting 3 for x, and 2 for y in (5), (6), or (7), and solving the resulting equation, we find

$$z = 1.$$

Substituting 3 for x, 2 for y, and 1 for z in (1), (2), (3), or (4), and solving the resulting equation, we find

$$t = 4.$$

By eliminating t from the given group, there results a second group, (5), (6), (7), containing one equation less, and by eliminating z from the second group, there results a third group, (8), (9), containing two equations less than the original group.

3. Let it be required to solve the equations

$$x + y + z = 26, \tag{1}$$
$$x - y = 4, \tag{2}$$
$$x - z = 6. \tag{3}$$

Adding (1), (2), (3), $3x = 36$;

whence, $x = 12$.

Hence, by (2) and (3), $y = 8$, and $z = 6$.

4. Let it be required to solve the equations

$$3x - 2y = 5, \quad (1)$$
$$2x + 2t = 8, \quad (2)$$
$$2x - 3t = 3, \quad (3)$$
$$4t + 3z = 7. \quad (4)$$

Combining (2) and (3), eliminating x,

$$5t = 5;$$

whence, $$t = 1.$$

Hence, by (2), $$2x + 2 = 8;$$

whence, $$x = 3.$$

Substituting 3 for x in (1), and 1 for t in (4), and solving the resulting equations, we find

$$y = 2, \quad \text{and} \quad z = 1.$$

5. Let it be required to solve the equations

$$2x + 7y = 29, \quad (1)$$
$$2t - 3x = 8, \quad (2)$$
$$3t - 2z = 26, \quad (3)$$
$$4x + 3z = 22. \quad (4)$$

Equations (2), (3), and (4) constitute a group of three equations containing only three unknown quantities, t, x, and z; hence the values of these unknown quantities may be found by solving this group.

Combining (2) and (3), eliminating t,

$$9x - 4z = 28. \quad (5)$$

Combining (4) and (5), eliminating z,

$$43x = 172;$$

whence, $$x = 4.$$

Substituting 4 for x in (1), (2), and (4), and solving the resulting equations, we find

$$y = 3, \quad t = 10, \quad \text{and} \quad z = 2.$$

RULE.

1. *By successive eliminations deduce from the given equations a single equation containing only one unknown quantity.*

2. *Find the value of this unknown quantity* (**161**).

3. *Having found the value of one of the unknown quantities, substitute it for that unknown quantity in any one of the equations containing that unknown quantity and only one other; then find the value of that other.*

4. *Having found the values of two unknown quantities, substitute them for those unknown quantities in any one of the equations containing them and only one other; then find the value of that other.*

5. *Continue this process until the values of all the unknown quantities are found.*

Cor. 1. If the number of unknown quantities is less than the number of independent equations, it may be impossible to satisfy all the equations; that is, they may be incompatible (**169**). Thus, the equations

$$x + y = 10, \qquad (1)$$

$$x - y = 8, \qquad (2)$$

$$2x + y = 11 \qquad (3)$$

cannot be satisfied by the same values of x and y; for solving (1) and (2), we find $x = 9$, and $y = 1$, and these values do not satisfy (3).

Cor. 2. If the number of unknown quantities exceeds the number of equations, there will be an infinite number of sets of values for the unknown quantities. Such equations are said to be *indeterminate.* Thus, the equations

$$x + y + 2z = 10,$$

$$5x + 2y - 2z = 20$$

are indeterminate, for eliminating z, we have

$$6x + 3y = 30,$$

and x and y, in this equation, may have an infinite number of values (**168**).

COR. 3. Hence the number of independent equations should be equal to the number of unknown quantities involved.

EXERCISES.

Solve the following groups of simultaneous equations:

1. $$\left.\begin{cases} x + 2y - z = 2 \\ 2x - y + 3z = 9 \\ 3x + y - 2z = -1 \end{cases}\right\}.$$

Ans. $x = 1$, $y = 2$, $z = 3$.

2. $$\left.\begin{cases} 3x - 2y - 2z = 2 \\ 5x - y + 3z = 23 \\ x - 5y + 6z = 1 \end{cases}\right\}.$$

Ans. $x = 4$, $y = 3$, $z = 2$.

3. $$\left.\begin{cases} x + y = 1 \\ x + z = 8 \\ y + z = -3 \end{cases}\right\}.$$

Ans. $x = 6$, $y = -5$, $z = 2$.

4. $$\left.\begin{cases} x + y + z = 12 \\ x + 2y - 2z = 10 \\ x + y - z = 4 \end{cases}\right\}.$$

Ans. $x = -2$, $y = 10$, $z = 4$.

5. $$\left.\begin{cases} x + 5y + 6z = 53 \\ x + 3y + 3z = 30 \\ x + y + z = 12 \end{cases}\right\}.$$

Ans. $x = 3$, $y = 4$, $z = 5$.

6. $$\left.\begin{cases} x + y + z = 12 \\ x + 2y + 3z = 20 \\ \frac{1}{3}x + \frac{1}{2}y + z = 6 \end{cases}\right\}.$$

Ans. $x = 6$, $y = 4$, $z = 2$.

7. $$\left.\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 62 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 47 \\ \frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 38 \end{cases}\right\}.$$

Ans. $x = 24,\ y = 60,\ z = 120.$

8. $$\left.\begin{cases} x - 2y = 2 \\ 3x + z = 28 \\ 2y - 3z = 14 \end{cases}\right\}.$$

Ans. $x = 10,\ y = 4,\ z = -2.$

9. $$\left.\begin{cases} 3x + 5y = 161 \\ 7x + 2z = 209 \\ 2y + z = 89 \end{cases}\right\}.$$

Ans. $x = 17,\ y = 22,\ z = 45.$

10. $$\left.\begin{cases} x + y = 9 \\ 3z - y = 3 \\ 3y + z = 11 \end{cases}\right\}.$$ *Ans.* $x = 6,\ y = 3,\ z = 2.$

11. $$\left.\begin{cases} x + y + z = 56 \\ x + y - z = 4 \\ x - y + z = 28 \end{cases}\right\}.$$

Ans. $x = 16,\ y = 14,\ z = 26.$

12. $$\left.\begin{cases} x - y + z = 0 \\ x - 2y + 3z = 1 \\ x - 3y + 4z = 0 \end{cases}\right\}.$$

Ans. $x = 1,\ y = 3,\ z = 2.$

13. $$\left.\begin{cases} 7x + 9y - 12z = 61 \\ 5x + 4y - 15z = 58 \\ 3x + 10y - 20z = 83 \end{cases}\right\}.$$

Ans. $x = 1,\ y = 2,\ z = -3.$

14. $$\left.\begin{cases} x + y + w = 9 \\ \frac{1}{2}y + z + \frac{1}{2}w = 8 \\ x + y + z = 12 \\ x + w + z = 10 \end{cases}\right\}.$$

Ans. $x = 3,\ y = 4,\ z = 5,\ w = 2.$

15. $$\left\{\begin{aligned} w + 50 &= x \\ x + 120 &= 3y \\ y + 120 &= 2z \\ z + 195 &= 3w \end{aligned}\right\}.$$

Ans. $w = 100,\ x = 150,\ y = 90,\ z = 105.$

16. $$\left\{\begin{aligned} 7u + 13z &= 87 \\ 10y - 3x &= 11 \\ 3u + 14x &= 57 \\ 2x + 11z &= 50 \end{aligned}\right\}.$$

Ans. $u = 5,\ x = 3,\ y = 2,\ z = 4.$

17. $$\left\{\begin{aligned} 7x + 2z + 3u &= 17 \\ 4y + 2z + v &= 11 \\ 5y - 3x - 2u &= 8 \\ 4y - 3u + 2v &= 9 \\ 8u - 3z &= 33 \end{aligned}\right\}.$$

Ans. $u = 3,\ v = 1,\ x = 2,\ y = 4,\ z = -3.$

18. $$\left\{\begin{aligned} x + y &= a \\ x + z &= b \\ y + z &= c \end{aligned}\right\}.$$

Ans. $x = \dfrac{a + b - c}{2},\quad y = \dfrac{a + c - b}{2},\quad z = \dfrac{b + c - a}{2}.$

19. $$\left\{\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 2 \\ \frac{x}{a} + \frac{z}{c} &= 2 \\ \frac{y}{b} + \frac{z}{c} &= 2 \end{aligned}\right\}.$$ *Ans.* $x = a,\ y = b,\ z = c.$

20. $$\left\{\begin{aligned} x + 11a &= y + z \\ y &= 2(x + z) - 11a \\ z &= 3(x + y) - 11a \end{aligned}\right\}.$$

Ans. $x = a,\ y = 5a,\ z = 7a.$

21. $$\left\{\begin{array}{l}\frac{1}{x}+\frac{2}{y}-\frac{3}{z}=1 \\ \frac{5}{x}+\frac{4}{y}+\frac{6}{z}=24 \\ \frac{7}{x}-\frac{8}{y}+\frac{9}{z}=14\end{array}\right\}.$$

Ans. $x=\frac{1}{2}$, $y=\frac{2}{3}$, $z=\frac{3}{4}$.

These equations are of the first degree, if $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$ are regarded as the unknown quantities.

PROBLEMS LEADING TO SIMPLE SIMULTANEOUS EQUATIONS.

178. In order that the unknown quantities of a problem may be found, it must furnish as many independent equations as there are unknown quantities (**177**, Cor. 3).

EXERCISES.

1. Find two numbers whose sum is 29, and difference 9.

Let $x =$ the greater number,

and $y =$ the less;

then, by the conditions of the problem,

$$x+y=29,$$

and $$x-y=9;$$

whence, $x=19$, and $y=10$.

2. Find two numbers whose sum is a, and whose difference is b.

Let $x =$ the greater number,

and $y =$ the less;

then, by the conditions of the problem,

$$x + y = a,$$

and $$x - y = b;$$

whence, $$x = \frac{a+b}{2}, \quad \text{and} \quad y = \frac{a-b}{2}.$$

Since a and b are general symbols, we have the following principles, by means of which all similar problems may be solved:

(1) *The greater of two numbers is found by adding half their difference to half their sum.*

(2) *The less of two numbers is found by subtracting half their difference from half their sum.*

3. Divide 29 into two such parts that three times the less shall exceed twice the greater by 2.

Let $x =$ the greater part;

and $y =$ the less;

then, by the conditions of the problem,

$$x + y = 29,$$

and $$3y - 2x = 2;$$

whence, $$x = 17, \quad \text{and} \quad y = 12.$$

4. If 27 be added to a certain number consisting of two digits, the order of the digits will be inverted; and one of the digits is two-thirds of the other; find the number.

Let $x =$ the digit in tens' place,

and $y =$ the digit in units' place;

then the number will be represented by $10x + y$.

By the first condition,

$$10x + y + 27 = 10y + x; \qquad (1)$$

whence, $$y - x = 3; \qquad (2)$$

$$\therefore \quad y > x.$$

The second condition requires one of the digits to be two-thirds of the other, and as $y > x$, we must have

$$x = \tfrac{2}{3}y. \tag{3}$$

Solving (2) and (3), we find

$$x = 6, \quad \text{and} \quad y = 9;$$

$\therefore \quad 10x + y = 60 + 9 = 69$, the number required.

5. Find a fraction such that, when 1 is added to its numerator, its value becomes $\frac{1}{3}$, and when 1 is added to its denominator, its value becomes $\frac{1}{4}$.

Let $x =$ the numerator,

and $y =$ the denominator of the required fraction;

then, by the conditions of the problem,

$$\frac{x+1}{y} = \frac{1}{3}, \tag{1}$$

and

$$\frac{x}{y+1} = \frac{1}{4}. \tag{2}$$

Clearing (1) and (2) of fractions,

$$3x + 3 = y, \tag{3}$$

$$4x = y + 1; \tag{4}$$

$$\therefore \quad 4x = 3x + 3 + 1;$$

whence, $x = 4.$

Substituting 4 for x in (3), we find

$$y = 15;$$

$\therefore \quad \frac{x}{y} = \frac{4}{15}$, the fraction required.

6. Find three numbers, such that their sums, taken two and two, shall be 12, 13, and 15.

Let the numbers be represented by x, y, and z; then, by the conditions of the problem,

$$x + y = 12, \tag{1}$$
$$y + z = 13, \tag{2}$$
$$x + z = 15. \tag{3}$$

Adding (1), (2), and (3),

$$2x + 2y + 2z = 40;$$

whence,
$$x + y + z = 20. \tag{4}$$

Subtracting (1) from (4), we find $z = 8$.

Subtracting (2) from (4), we find $x = 7$.

Subtracting (3) from (4), we find $y = 5$.

7. If 5 lbs. of tea and 3 lbs. of coffee cost $3.50, and 8 lbs. of tea and 6 lbs. of coffee cost $6, what is the price per pound of each?

Let $x =$ the price of 1 lb. of tea, in cents,

and $y =$ the price of 1 lb. of coffee, in cents;

then, by the conditions of the problem,

$$5x + 3y = 350,$$

and
$$8x + 6y = 600;$$

whence, $x = 50$, and $y = 33\frac{1}{3}$.

8. Nine gallons of a mixture of fresh and sea water are found to weigh $91\frac{1}{4}$ lbs.; assuming that a gallon of fresh water weighs 10 lbs., and that the specific gravity of sea water is 1.025, find the number of gallons of each in the mixture.

Let $x =$ the number of gallons of fresh water,

and $y =$ the number of gallons of sea water;

then, since the mixture contains 9 gallons,

$$x + y = 9.$$

The weight of x gallons of fresh water is $10x$ lbs., and the weight of y gallons of sea water is $10y \times 1.025$ lbs.;

$$\therefore \quad 10x + 10y \times 1.025 = 91\tfrac{1}{4}.$$

Solving these equations, we find

$$x = 4, \quad \text{and} \quad y = 5.$$

9. Find two numbers such that, if the first be added to three times the second, the sum will be 23; and if the second be added to five times the first, the sum will be 31.
Ans. 5 and 6.

10. If A's money were diminished by \$5, he would have half as much as B; but, if B's money were increased by \$36, he would have three times as much as A: how much has each? *Ans.* A, \$26; B, \$42.

11. If A gives \$20 to B, they will then have equal amounts; but, if B gives \$15 to A, B will then have two-ninths as much as A: how much money has each?
Ans. A, \$75; B, \$35.

12. For \$3.70, 5 lbs. of tea and 4 lbs. of coffee can be bought; but, if tea were to rise 20 per cent., and coffee, 10 per cent., they would cost \$4.32: find the price per pound of each. *Ans.* Tea, 50 cts.; coffee, 30 cts.

13. Seven years ago, the age of A was three times that of B; and seven years hence, the age of A will be twice that of B: find their present ages.
Ans. A's, 49 years; B's, 21 years.

14. If a certain number consisting of two digits be multiplied by 2, and the product diminished by 4, the order of the digits will be inverted; and if 10 be subtracted from the number, the remainder will be equal to three times the sum of the digits: find the number. *Ans.* 49.

15. A has four-ninths as much money as B; but, if B gives \$10 to A, they will have equal amounts: how much has each? *Ans.* A, \$16; B, \$36.

16. A bill of 745 francs was paid in five-franc and two-franc pieces, and the whole number of pieces was 185: how many were there of each?

Ans. 125 five-franc pieces, and 60 two-franc pieces.

17. Three men and two boys working together one day dug 195 feet of a garden border; on another day, two men and three boys dug 180 feet: how many feet did each dig in a day? *Ans.* One man, 45 feet; one boy, 30 feet.

18. Find a fraction such that, if its numerator be doubled, and its denominator be increased by 3, the result will be equal to $\frac{1}{3}$; but, if its denominator be doubled, and its numerator be increased by 7, the result will be equal to $\frac{1}{2}$.

Ans. $\frac{2}{9}$.

19. A sum of money was divided equally among a certain number of persons; if there had been six more persons, each would have received \$2 less than he did; and if the number of persons had been three less, each would have received \$2 more than he did: find the number of persons, and what each received. *Ans.* 12, and each received \$6.

20. A certain number consisting of two digits is equal to five times the sum of its digits; and if 9 be added to the number, the order of the digits will be inverted: find the number. *Ans.* 45.

21. Two trains, one of which is 92 feet long and the other 84 feet long, are moving with uniform velocities on parallel tracks; when they move in opposite directions, they pass each other in one second and a half; but when they move in the same direction, the faster train passes the other in 6 seconds: find the rate at which each train moves.

Ans. One 30, the other 50 miles per hour.

22. Hiero's crown, which was made of gold and silver, is said to have weighed 20 lbs., which was 16 times the weight of an equal volume of water; assuming the specific gravity of gold to be $19\frac{1}{4}$, and that of silver to be $10\frac{1}{2}$, find the weight of each metal.

Ans. $15\frac{1}{8}$ lbs. of gold; $4\frac{7}{8}$ lbs. of silver.

23. A railway train, after traveling for one hour, is detained 24 minutes, after which it proceeds at six-fifths of its former rate, and arrives at the terminus 15 minutes late. If the detention had occurred 5 miles further on, the train would have reached the terminus 2 minutes later than it did. Find the length of the line, and the original rate of the train. *Ans.* { Length of the line, $47\frac{1}{2}$ miles; original rate, 25 miles per hour.

24. A and B ran a race which lasted 5 minutes; B had a start of 20 rods; but A ran 3 yards while B ran 2, and won by 30 rods: find the length of the course and the speed of each. *Ans.* { Length of course, 150 rods; A ran 30, and B 20 rods per minute.

25. Two passengers have together 300 lbs. of baggage, and one of them has to pay $1.60, and the other $0.40 for the excess above the weight allowed; if all the baggage had belonged to one of them, he would have had to pay $4: find how much baggage each passenger is allowed without charge. *Ans.* 100 lbs.

26. A and B together can perform a piece of work in 32 days, B and C, in 120 days, and A, B, and C, in 30 days: in what time could each alone perform the work?

Ans. A, in 40 days; B, in 160 days; C, in 480 days.

27. A, B, and C together can perform a piece of work in 15 days; A and B together perform four-thirds of what C does; and C does twice as much work as A: in what time could each alone perform the work?

Ans. A, in 70 days; B, in 42 days; C, in 35 days.

28. A cistern holding 1200 gallons can be filled by three pipes, A, B, and C together, in two-fifths of an hour. The pipe A requires half an hour more than C to fill the cistern; and 10 gallons less run through C per minute than through A and B together. Find the time in which each pipe alone could fill the cistern.

Ans. A, in 90 min.; B, in 72 min.; C, in 60 min.

DISCUSSION OF PROBLEMS.

179. An **Arbitrary Quantity** is a quantity to which any value whatever may be assigned. Thus, in the problem, "To find two numbers whose sum is a, and whose difference is b," a and b are the arbitrary quantities.

180. The **Discussion** of a problem consists in making different suppositions with regard to the *arbitrary* quantities involved, and interpreting the results.

181. *A's age is* a *years, and B's age is* b *years: when will the age of A be twice that of B?*

Let $x =$ the required number of years;

then, by the problem,

$$a + x = 2(b + x);$$

whence,

$$x = a - 2b.$$

DISCUSSION.

1. Suppose $a = 40$, and $b = 16$;

then $x = 40 - 32 = 8$;

that is, if A's age is 40 years, and B's age is 16 years, A *will* be twice as old as B in 8 years.

2. Suppose $a = 40$, and $b = 20$;

then $x = 40 - 40 = 0$.

This result is as it ought to be; for, if A is 40 years old, and B is 20 years old, A is twice as old as B *now*.

3. Suppose $a = 35$, and $b = 20$;

then $x = 35 - 40 = -5$.

This negative result shows that A will never be twice as old as B, but that he *was* twice as old 5 years ago (**68**, Sch. 1).

For these values of a and b, therefore, the problem ought to be stated as follows:

A's age is 35 years, and B's age is 20 years: when *was* the age of A twice that of B?

Let $x =$ the required number of years;

then, by the problem,

$$35 - x = 2(20 - x);$$

whence, $x = 5$.

It should be observed that the equation corresponding to the new enunciation may be obtained from the original equation by simply changing the sign of x.

4. Suppose the problem had been enunciated as follows:

A's age is a years, and B's age is b years: find the epoch at which A's age is twice that of B.

From this enunciation it cannot be decided whether the required epoch is in the future or in the past. If we suppose it to be x years *after* the present date, we shall find, as before,

$$x = a - 2b;$$

but if we suppose it to be x years before the present date, we shall find the algebraic statement to be

$$a - x = 2(b - x);$$

whence, $x = 2b - a$.

If $2b < a$, the first supposition is correct, since it leads to a positive value for x.

If $2b > a$, the second supposition is correct, since it leads to a positive value for x.

In this case, then, a negative result would indicate that we made the wrong choice out of two possible suppositions.

It should be observed, however, that, when we find we have made the wrong choice, it is not necessary to go through the whole investigation again; for we can use the result obtained on the wrong supposition. We have simply to take the absolute value of the negative result and place the epoch in the *past*, if we had supposed it in the future, or in the *future*, if we had supposed it in the past.

182. *What number must be added to both terms of the fraction* $\frac{a}{b}$, *in order that the result shall be equal to* m?

Let $x =$ the required number;

then, by the problem, $\frac{a+x}{b+x} = m$;

whence, $$x = \frac{bm - a}{1 - m}.$$

DISCUSSION.

1. Suppose $\frac{a}{b} = \frac{4}{7}$, and $m = \frac{2}{3}$;

then $$x = \frac{7 \times \frac{2}{3} - 4}{1 - \frac{2}{3}} = \frac{\frac{14}{3} - 4}{\frac{1}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2.$$

This result indicates that, if 2 be added to both terms of $\frac{4}{7}$, the result will be equal to $\frac{2}{3}$. Thus,

$$\frac{4+2}{7+2} = \frac{6}{9} = \frac{2}{3}.$$

2. Suppose $\frac{a}{b} = \frac{5}{8}$, and $m = \frac{1}{2}$;

then $$x = \frac{8 \times \frac{1}{2} - 5}{1 - \frac{1}{2}} = \frac{4 - 5}{\frac{1}{2}} = \frac{-1}{\frac{1}{2}} = -2.$$

This negative result shows that 2 must be *subtracted* from both terms of the fraction $\frac{5}{8}$, in order that the result shall be equal to $\frac{1}{2}$. Thus, $\frac{5-2}{8-2} = \frac{3}{6} = \frac{1}{2}$. For these values of $\frac{a}{b}$ and m, therefore, the problem ought to be stated as follows:

What number must be *subtracted* from both terms of the fraction $\frac{5}{8}$, in order that the result shall be equal to $\frac{1}{2}$?

The algebraic statement of this problem is

$$\frac{5-x}{8-x} = \frac{1}{2};$$

whence,

$$x = 2.$$

3. Suppose $\frac{a}{b} = \frac{5}{9}$, and $m = 1$;

then

$$x = \frac{9-5}{1-1} = \frac{4}{0}.$$

How is this result to be interpreted?

The value of a fraction whose numerator remains unchanged *increases*, when its denominator *decreases*. Thus,

$$\frac{4}{2} = 2, \quad \frac{4}{1} = 4, \quad \frac{4}{\frac{1}{2}} = 8, \quad \frac{4}{\frac{1}{10}} = 40,$$

$$\frac{4}{\frac{1}{100}} = 400, \quad \frac{4}{\frac{1}{1000000}} = 4000000.$$

Therefore, if the denominator is infinitely small, or equal to 0 (zero), the value of the fraction is infinitely great. Hence, in this case, the value of x is infinitely great. Therefore, for these values of $\frac{a}{b}$ and m, the problem is impossible.

That this interpretation is correct is evident; for the equation for this case is $\frac{5+x}{9+x} = 1$, which is impossible, since the numerator is not equal to the denominator.

Hence, a value of the form of $\frac{a}{0}$, $\frac{4}{0}$, $\frac{1}{0}$, etc., indicates that the problem is impossible.

A quantity infinitely large is sometimes represented by the symbol ∞. Hence, $\frac{a}{0} = \infty$.

4. Suppose $\frac{a}{b} = \frac{5}{5}$, and $m = 1$;

then $$x = \frac{5-5}{1-1} = \frac{0}{0}.$$

How is this result to be interpreted?

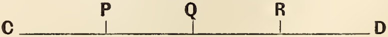
The value of x in this case is the quotient of zero divided by zero; that is, x is equal to a number which, when multiplied by zero, gives a product equal to zero. But *any* finite number multiplied by zero gives zero. Thus,

$$2 \times 0 = 0, \quad 3 \times 0 = 0, \quad 4 \times 0 = 0, \quad 25 \times 0 = 0, \quad \text{etc.}$$

Hence, x has an infinite number of values; in other words, the problem is indeterminate.

That this interpretation is correct is evident; for the equation for this case is $\frac{5+x}{5+x} = 1$, which is satisfied for any value that may be assigned to x.

183. *Two couriers, A and B, were traveling along the same road from C toward D; the former going at the rate of* m *miles per hour, and the latter, at the rate of* n *miles per hour. At 12 o'clock, A was at P, and B was at Q,* d *miles in advance of A. When were the couriers together?*



We cannot decide, from the enunciation of the problem, whether the couriers were together before or after 12 o'clock; but in order to put the problem into an equation, let us suppose the required time to be *after* 12 o'clock. We must then regard time after 12 o'clock as positive, and time before 12 o'clock as negative.

Let R be the point where the couriers were together, and let $x =$ the required number of hours; then, since A traveled at the rate of m miles per hour, and B at the rate of n miles per hour,

$$PR = mx, \quad \text{and} \quad QR = nx.$$

But $$PR = PQ + QR;$$

$$\therefore \quad mx = d + nx;$$

whence, $$x = \frac{d}{m - n}.$$

DISCUSSION.

1. Suppose $m > n$; then the value of x will be positive.

This positive value of x indicates that the couriers were together *after* 12 o'clock.

This interpretation is evidently correct; for, if $m > n$, A was traveling faster than B. Hence, A would gain upon B, and overtake him some time after 12 o'clock.

2. Suppose $m < n$; then the value of x will be negative.

This negative value of x indicates that the couriers were together *before* 12 o'clock.

This interpretation is evidently correct; for, if $m < n$, B was traveling faster than A; and, as B was in advance of A at 12 o'clock, he must have passed A before that time.

3. Suppose $m = n$; then

$$x = \frac{d}{0} = \infty.$$

Hence, in this case the problem is impossible.

This interpretation is in accordance with the conditions of the problem; for, at 12 o'clock the couriers were d miles apart; and, if $m = n$, they were traveling at equal rates; hence they would never come together.

4. Suppose $d = 0$, and $m > n$, or $m < n$; then

$$x = \frac{0}{m - n} = 0.$$

This result indicates that the couriers were together at 12 o'clock, and at no other time.

This interpretation is evidently the true one; for, if $d = 0$, then, at 12 o'clock, A and B were together at the point P; and, if $m > n$, or $m < n$, they were traveling at different rates, and must have been either approaching or receding from each other at all other times.

5. Suppose $d = 0$, and $m = n$; then

$$x = \frac{0}{0}.$$

This result indicates that the couriers were together all the time, for $\frac{0}{0}$ is equal to any finite quantity (**182**, 4).

This interpretation is confirmed by the conditions of the problem; for, if $d = 0$, the couriers were together at 12 o'clock; and, if $m = n$, they were traveling at equal rates, and would never separate.

CHAPTER VIII.

INEQUALITIES.

184. If the difference between two quantities is *positive*, the minuend is greater than the subtrahend; but, if the difference is *negative*, the minuend is less than the subtrahend. Thus, if $a - b$ is positive, $a > b$; but, if $a - b$ is negative, $a < b$.

185. Two inequalities *subsist in the same sense*, if the first member is the greater in each, or the less in each. Thus, $6 > 5$ and $9 > 7$ subsist in the same sense.

Two inequalities *subsist in a contrary sense*, if the first member is the greater in one, and the less in the other. Thus, $4 > 3$ and $2 < 5$ subsist in a contrary sense.

186. TH. *If the same quantity be added to or subtracted from each member of an inequality, the resulting inequality will subsist in the same sense.*

Suppose $a > b$; then $a - b$ is positive (**184**).

Since $$a + c - (b + c) = a - b,$$

and $$a - c - (b - c) = a - b,$$

it follows that $a + c - (b + c)$ and $(a - c) - (b - c)$ are positive;

$$\therefore \quad a + c > b + c,$$

and $$a - c > b - c.$$

COR. 1. The rule for the transposition of terms in an equation is applicable to inequalities. Thus, if

$$a^2 - 2ab + b^2 > c^2;$$

then, by the theorem,

$$a^2 - 2ab + b^2 + 2ab > c^2 + 2ab,$$

or

$$a^2 + b^2 > c^2 + 2ab.$$

Cor. 2. If an equation be added to an inequality, member to member, or subtracted from it, member from member, the resulting inequality will subsist in the same sense.

Thus, if $a > b,$

and $c = d;$

then $a + c > b + d,$

and $a - c > b - d.$

187. Th. *If an inequality be subtracted from an equation, member from member, the resulting inequality will subsist in a contrary sense.*

Suppose $c = d,$

and $a > b;$

then $c - a - (d - b) = b - a.$

But $b - a$ is negative, for $a > b$; hence, $c - a - (d-b)$ is negative; that is, $c - a < d - b.$

188. Th. *If both members of an inequality be multiplied or divided by the same positive quantity, the resulting inequality will subsist in the same sense.*

Suppose m to be positive, and

$$a > b;$$

then, since $a - b$ is positive, $ma - mb$ and $\frac{a}{m} - \frac{b}{m}$ are positive (**77**, Cor. 1; **84**, Cor.);

$$ma > mb,$$

and

$$\frac{a}{m} > \frac{b}{m}.$$

Cor. The rule for clearing an equation of fractions is applicable to an inequality.

189. TH. *If both members of an inequality be multiplied or divided by the same negative quantity, the resulting inequality will subsist in a contrary sense.*

Suppose m to be negative, and

$$a > b;$$

then, since $a - b$ is positive, $ma - mb$ and $\frac{a}{m} - \frac{b}{m}$ are negative;

$$\therefore \quad ma < mb,$$

and $$\frac{a}{m} < \frac{b}{m}.$$

COR. If the sign of each term of an inequality be changed, the sign of inequality must be reversed; for changing the sign of each term is equivalent to multiplying both members by -1.

190. TH. *If two or more inequalities subsisting in the same sense be added, member to member, the resulting inequality will subsist in the same sense as the given inequalities.*

Suppose $\quad a > b, \quad c > d, \quad$ and $\quad e > f;$

then $a - b$, $c - d$, and $e - f$ are positive; hence their sum, $a + c + e - (b + d + f)$, is positive; that is,

$$a + c + e > b + d + f.$$

191. TH. *If one inequality be subtracted from another subsisting in the same sense, the result will, in some cases, be an inequality subsisting in the same sense as the given inequalities, in some, an inequality subsisting in a contrary sense, and, in some, an equation.*

1. Take the inequalities

$$7 > 4,$$

and $$\underline{3 > 2.}$$

By subtraction, $$4 > 2.$$

The result in this case is an inequality subsisting in the same sense as the given inequalities.

2. Take the inequalities

$$\begin{array}{lr} & 10 > 9, \\ \text{and} & 8 > 6. \\ \hline \text{By subtraction,} & 2 < 3. \end{array}$$

The result in this case is an inequality subsisting in a sense *contrary* to that of the given inequalities.

3. Take the inequalities

$$\begin{array}{lr} & 10 > 9, \\ \text{and} & 7 > 6. \\ \hline \text{By subtraction,} & 3 = 3. \end{array}$$

The result in this case is an equation.

SOLUTION OF INEQUALITIES

CONTAINING ONLY ONE UNKNOWN QUANTITY.

192. The **Solution** of an inequality containing only one unknown quantity consists in transforming it in such a manner that the unknown quantity shall stand alone as one of its members. The other member will then be one *limit* of the unknown quantity.

EXERCISES.

1. Solve the inequality

$$\frac{x}{2} + \frac{2x}{5} > \frac{3x}{4} + 3.$$

Clearing of fractions,

$$10x + 8x > 15x + 60;$$

transposing and reducing,

$$3x > 60;$$

whence, $x > 20.$

Solve the following inequalities:

2. $\frac{2x}{5} - \frac{3x}{4} > \frac{9}{4} - \frac{x}{2}.$ *Ans.* $x > 15.$

3. $5x + 2 > \frac{3x}{2} + 16.$ *Ans.* $x > 4.$

4. $2 - \frac{2x}{3} > 0.$ *Ans.* $x < 3.$

5. $\frac{7x}{12} + \frac{11}{6} > \frac{5x}{8} + \frac{5}{4}.$ *Ans.* $x < 14.$

6. $3x - \frac{x+2}{4} < \frac{2x+3}{5}.$ *Ans.* $x < \frac{22}{47}.$

AN EQUATION AND AN INEQUALITY COMBINED.

193. If there be given an equation and an inequality containing only two unknown quantities, a limit of each unknown quantity may be found.

EXERCISES.

1. Find a limit for x and y in the group,

$$2x + 5y > 32, \qquad (1)$$

$$2x + y = 24. \qquad (2)$$

Subtracting (2) from (1), $4y > 8;$

whence, $y > 2.$

If 2 be substituted for y in (2), the first member will be made less than the second;

$$\therefore\quad 2x + 2 < 24;$$

whence, $$x < 11.$$

Find a limit for x and y in each of the following groups:

2. $\left\{\begin{array}{r} x + 2y > 15 \\ 3x + 2y = 31 \end{array}\right\}.$ *Ans.* $x < 8,\ y > 3\frac{1}{2}$.

3. $\left\{\begin{array}{l} x - \dfrac{3y}{5} < \ 3 \\ \dfrac{9}{2}x + y = 23 \end{array}\right\}.$ *Ans.* $x < 4\frac{20}{37},\ y > 2\frac{21}{37}$.

4. $\left\{\begin{array}{r} 10y + 59 > 7x \\ x + \dfrac{5y}{4} = 17 \end{array}\right\}.$ *Ans.* $x < 13,\ y > 3\frac{1}{5}$.

5. $\left\{\begin{array}{r} 5x + 3y > 121 \\ 1\frac{3}{4}x + \ y = \ 42 \end{array}\right\}.$ *Ans.* $x < 20,\ y > 7$.

6. $\left\{\begin{array}{l} \dfrac{x+1}{y} > \dfrac{1}{2} \\ 3x = y + 3 \end{array}\right\}.$ *Ans.* $x < 5,\ y < 12$.

7. $\left\{\begin{array}{l} \dfrac{1}{x} + \dfrac{1}{y} > \dfrac{8}{15} \\ \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{2}{15} \end{array}\right\}.$ *Ans.* $x < 3,\ y < 5$.

CHAPTER IX.

INVOLUTION AND EVOLUTION.

INVOLUTION.

194. **Involution** is the process of finding any power of a quantity.

195. The **Exponent of a Power** is the exponent which indicates the number of times the base must be used to produce that power. Thus, in the power a^3, 3 is the exponent of the power.

196. A **Perfect Power** is an expression which can be resolved into as many equal factors as there are units in the degree of the power. Thus, $a^2 + 2ab + b^2$ is a perfect second power, or square; for $a^2 + 2ab + b^2 = (a + b)(a + b)$.

197. An **Imperfect Power** is an expression which cannot be resolved into as many equal factors as there are units in the degree of the power.

An expression may be a *perfect* power of one degree, but an *imperfect* power of another degree. Thus, $a^2 + 2ab + b^2$ is a perfect square, but an imperfect cube.

198. TH. *Any power of a positive quantity is positive* (**77**).

199. TH. *Any even power of a negative quantity is positive; and any odd power of a negative quantity is negative* (**77**, Cor. 2).

Thus, $(-a)^2 = +a^2$, $(-a)^3 = -a^3$, $(-a)^4 = +a^4$, and so on.

200. TH. *The n^{th} power of the product of two or more factors is equal to the product of their n^{th} powers.*

The truth of this theorem may be made to appear by considering the following identities (**143**):

$$(ab)^2 = (ab)\,(ab) = a^2b^2;$$
$$(ab)^3 = (ab)\,(ab)\,(ab) = a^3b^3;$$
$$(abc)^3 = (abc)\,(abc)\,(abc) = a^3b^3c^3;$$
$$(abc)^4 = (abc)\,(abc)\,(abc)\,(abc) = a^4b^4c^4;$$
$$(abc)^n = (abc)\,(abc)\,(abc) \ldots\ldots \text{ to } n \text{ factors} = a^nb^nc^n.$$

201. TH. *The m^{th} power of the n^{th} power of a quantity is equal to the mn^{th} power of that quantity.*

This theorem may be proved as follows·

$$(a^3)^2 = a^3 \times a^3 = a^6;$$
$$(a^n)^2 = a^n \times a^n = a^{2n};$$
$$(a^n)^3 = a^n \times a^n \times a^n = a^{3n};$$
$$(a^n)^4 = a^n \times a^n \times a^n \times a^n = a^{4n};$$
$$(a^n)^m = a^n \times a^n \times a^n \times \ldots\ldots \text{ to } m \text{ factors} = a^{mn}.$$

202. To find any power of a monomial.

1. Let it be required to find the third power of $5a^2b^3$.

$$(5a^2b^3)^3 = (5a^2b^3)\,(5a^2b^3)\,(5a^2b^3) = 5^3a^{2\times3}b^{3\times3} = 125a^6b^9.$$

2. Let it be required to find the fifth power of $-3a^2b^3$.

$$(-3a^2b^3)^5 = (-3a^2b^3)\,(-3a^2b^3)\,(-3a^2b^3)\,(-3a^2b^3)\,(-3a^2b^3)$$
$$= (-3)^5a^{2\times5}b^{3\times5} = -243a^{10}b^{15}.$$

RULE.

Raise the numerical coefficient to the required power, and annex to the result all the letters of the given monomial, giving to each an exponent equal to the product of its original exponent by the exponent of the power.

EXERCISES.

1. Find the 3d power of a^9. *Ans.* a^{27}.

2. Find the 3d power of $-a^2b^3$. *Ans.* $-a^6b^9$.

3. Find the 4th power of $-2a^3b^2c$. *Ans.* $16a^{12}b^8c^4$.

4. Find the 5th power of $-2a^3b^2c$. *Ans.* $-32a^{15}b^{10}c^5$.

5. Find the 7th power of $-x^2y^3z^4$. *Ans.* $-x^{14}y^{21}z^{28}$.

6. Find the n^{th} power of $-a^2b^3c$ when n is an even positive integer. *Ans.* $a^{2n}b^{3n}c^n$.

7. Find the n^{th} power of $-a^2b^3c$ when n is an odd positive integer. *Ans.* $-a^{2n}b^{3n}c^n$.

8. Show that $[(a^m)^n]^p = a^{mnp}$.

203. To find any power of a polynomial.

RULE.

Find the product of as many factors, each equal to the given polynomial, as there are units in the exponent of the power.

COR. 1. Since $(a^n)^m = (a^m)^n$ (**201**), the same result may be reached by different processes. Thus, the 6th power of $x + a$ may be found, (1) by squaring $x + a$, then multiplying the result by $x + a$, and so on until $x + a$ has been used as a factor six times; (2) by finding the cube of $x + a$, and squaring the result, (3) by finding the square of $x + a$, and cubing the result.

COR. 2. The process of involution may sometimes be abridged by means of the principle expressed by the equation $x^m x^n = x^{m+n}$. Thus, the 5th power of $x + a$ may be found by multiplying the cube of $x + a$ by the square of $x + a$.

COR. 3. It may be shown by actual multiplication, that $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2\,(ab + ac + ad + bc + bd + cd)$.

Hence,

The square of a polynomial consists of two parts: (1) *the sum of the squares of its terms;* (2) *twice the sum of the products obtained by multiplying each term by the sum of all the terms which follow it.*

EXERCISES.

1. Find the square of $x - a + c$.
Ans. $x^2 + a^2 + c^2 - 2ax + 2cx - 2ac$.

2. Find the square of $x + a - b + c$ by the principle of Cor. 3.
Ans. $x^2 + a^2 + b^2 + c^2 + 2ax - 2bx + 2cx - 2ab + 2ac - 2bc$.

3. Find the 3d power of $a + b + c$.
Ans. $a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3ac^2 + 3bc^2 + 6abc$.

4. Find the 4th power of $x + a$.
Ans. $x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$.

5. Find the 6th power of $x - a$ by the principle of Cor. 1.
Ans. $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6$.

6. Find the 7th power of $x + a$ by the principle of Cor. 2.
Ans. $x^7 + 7ax^6 + 21a^2x^5 + 35a^3x^4 + 35a^4x^3 + 21a^5x^2 + + 7a^6x + a^7$.

204. To find any power of a fraction.

$$\left(\frac{x}{a}\right)^2 = \frac{x}{a} \times \frac{x}{a} = \frac{x^2}{a^2};$$

$$\left(\frac{x}{a}\right)^3 = \frac{x^2}{a^2} \times \frac{x}{a} = \frac{x^3}{a^3};$$

$$\left(\frac{x}{a}\right)^4 = \frac{x^3}{a^3} \times \frac{x}{a} = \frac{x^4}{a^4};$$

$$\left(-\frac{a^3}{b^2}\right)^3 = \left(-\frac{a^3}{b^2}\right)\left(-\frac{a^3}{b^2}\right)\left(-\frac{a^3}{b^2}\right) = -\frac{a^9}{b^6}.$$

RULE.

Raise both terms of the given fraction to the power whose exponent is equal to that of the required power, and give the proper sign to the result (**198, 199**).

EXERCISES.

1. Find the square of $-\frac{4x}{3y^2}$. *Ans.* $\frac{16x^2}{9y^4}$.

2. Find the 4th power of $-\frac{x^3}{y^2z^2}$. *Ans.* $\frac{x^{12}}{y^8z^8}$.

3. Find the square of $\frac{a-b}{a+b}$ *Ans.* $\frac{a^2-2ab+b^2}{a^2+2ab+b^2}$.

4. Find the cube of $\left(\frac{-ab}{c^2d}\right)^2$. *Ans.* $\frac{a^6b^6}{c^{12}d^6}$.

5. Find the 4th power of $\pm\frac{a^2bc}{4d^2x}$. *Ans.* $\frac{a^8b^4c^4}{256d^8x^4}$.

6. Find the 5th power of $\frac{(-ab^2c)^3}{-2mn}$. *Ans.* $\frac{a^{15}b^{30}c^{15}}{32m^5n^5}$.

7. Find the square of $\frac{x+y}{x^3y^4}$. *Ans.* $\frac{x^2+2xy+y^2}{x^6y^8}$.

8. Find the n^{th} power of $\frac{2abc}{3m^2n^3}$. *Ans.* $\frac{2^na^nb^nc^n}{3^nm^{2n}n^{3n}}$.

THE BINOMIAL THEOREM.

205. By means of the Binomial Theorem the operation of finding the higher powers of binomials may be very much abridged.

It may be shown by actual multiplication that

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5,$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

By an examination of these identities (**143**) it will be observed that in each of them

1. *The number of terms in the second member is one greater than the exponent in the first member; that is, the number of terms in the power is one greater than the exponent of the power.*

2. *The exponent of* a *in the first term of the power is equal to the exponent of the power, and in each of the succeeding terms the exponent of* a *is one less than in the preceding term.*

3. *The exponent of* b *in the second term of the power is 1, and in each of the succeeding terms the exponent of* b *is one greater than in the preceding term.*

4. *The sum of the exponents of* a *and* b *in any term of the power is equal to the exponent of the power.*

5. *The coefficient of the first term of the power is 1; the coefficient of the second term is equal to the exponent of the power; the coefficient of the third term is equal to the result*

obtained by multiplying the exponent of a in the second term by the coefficient of that term and dividing the result by 2; and, in general, if the exponent of a in any term be multiplied by the coefficient of that term and the result be divided by the number of that term, the quotient will be the coefficient of the succeeding term.

6. *The coefficients of any two terms of the power equally distant from the extremes are equal.*

THE BINOMIAL FORMULA.

206. From the principles of the preceding article it follows that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\times 3}a^{n-3}b^3 + \ldots\ldots + b^n.$$

This is the *Binomial Formula.* The second member of this formula is called the *Expansion*, or *Development*, of $(a+b)^n$.

207. The expansion of $(a-b)^n$ may be obtained by putting $-b$ in the place of $+b$ in the expansion of $(a+b)^n$. The terms containing the even powers of $-b$ will be positive, and those containing the odd powers of $-b$ will be negative.

$$\therefore\ (a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 - n\frac{(n-1)(n-2)}{2\times 3}a^{n-3}b^3 + \ldots\ldots \pm b^n.$$

The last term of the expansion of $(a-b)^n$ is positive or negative, according as n, the exponent of the power, is even or odd.

208. To find any power of a binomial by means of the binomial theorem.

EXERCISES.

1. Find the 4th power of $c + x$.

First Solution. The first term of the expansion is c^4 (**205**, 2, 5); the second term is $4c^3x$ (**205**, 3, 5);

$$\text{the third term} = \frac{4 \times 3c^2x^2}{2} = 6c^2x^2;$$

$$\text{the fourth term} = \frac{6 \times 2cx^3}{3} = 4cx^3;$$

$$\text{the fifth term} = \frac{4 \times 1x^4}{4} = x^4.$$

$$\therefore \quad (c + x)^4 = c^4 + 4c^3x + 6c^2x^2 + 4cx^3 + x^4.$$

Second Solution. The literal parts of the terms of the expansion of $(c + x)^4$ are

$$\begin{array}{lccccc} & c^4, & c^3x, & c^2x^2, & cx^3, & x^4, \\ \text{and the coefficients are} & 1, & 4, & 6, & 4, & 1; \\ \hline \therefore \quad (c + x)^4 = & c^4 + & 4c^3x + & 6c^2x^2 + & 4cx^3 + & x^4. \end{array}$$

Third Solution. Substitute c for a, x for b, and 4 for n in the Binomial Formula; then

$$(c + x)^4 = c^4 + 4c^3x + 6c^2x^2 + 4cx^3 + x^4.$$

2. Find the 4th power of $2x - 3a$.

Substitute $2x$ for c, and $-3a$ for x in the equation

$$(c + x)^4 = c^4 + 4c^3x + 6c^2x^2 + 4cx^3 + x^4,$$

$$\begin{aligned} \text{then} \quad (2x - 3a)^4 &= (2x)^4 + 4(2x)^3(-3a) + 6(2x)^2(-3a)^2 \\ &\quad + 4(2x)(-3a)^3 + (-3a)^4 \\ &= 16x^4 - 96ax^3 + 216a^2x^2 - 216a^3x + 81a^4. \end{aligned}$$

The same result may be obtained by the following method:

$$\begin{array}{llllll}
\text{Powers of } 2x: & (2x)^4, & (2x)^3, & (2x)^2, & (2x)^1; & \\
\text{Powers of } -3a: & & (-3a)^1, & (-3a)^2, & (-3a)^3, & (-3a)^4; \\
\text{Coefficients:} & 1, & 4, & 6, & 4, & 1; \\
\hline
\end{array}$$

$$\therefore\ (2x-3a)^4 = 16x^4 - 96ax^3 + 216a^2x^2 - 216a^3x + 81a^4.$$

Expand:

3. $(x-a)^4$. *Ans.* $x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$.

4. $(x+a)^5$.
Ans. $x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5$.

5. $(x+a)^7$.
Ans. $x^7 + 7ax^6 + 21a^2x^5 + 35a^3x^4 + 35a^4x^3 + 21a^5x^2 + 7a^6x + a^7$.

6. $(2x+3a)^5$.
Ans. $32x^5 + 240ax^4 + 720a^2x^3 + 1080a^3x^2 + 810a^4x + 243a^5$.

7. $(2-x)^3$. *Ans.* $8 - 12x + 6x^2 - x^3$.

8. $(x-7)^4$. *Ans.* $x^4 - 28x^3 + 294x^2 - 1372x + 2401$.

9. $(2a-b)^5$.
Ans. $32a^5 - 80a^4b + 80a^3b^2 - 40a^2b^3 + 10ab^4 - b^5$.

10. $(2a-7c)^7$.
Ans. $128a^7 - 3136a^6c + 32928a^5c^2 - 192080a^4c^3 + 672280a^3c^4 - 1411788a^2c^5 + 1647086ac^6 - 823543c^7$.

11. $(1-x)^9$.
Ans. $1 - 9x + 36x^2 - 84x^3 + 126x^4 - 126x^5 + 84x^6 - 36x^7 + 9x^8 - x^9$.

12. $\left(2 - \frac{1}{y}\right)^4$. *Ans.* $16 - \frac{32}{y} + \frac{24}{y^2} - \frac{8}{y^3} + \frac{1}{y^4}$.

13. $\left(1 - \frac{2}{3x}\right)^6$.

Ans. $1 - \frac{4}{x} + \frac{20}{3x^2} - \frac{160}{27x^3} + \frac{80}{27x^4} - \frac{64}{81x^5} + \frac{64}{729x^6}$.

14. $(5a - 7b)^4$.

Ans. $625a^4 - 3500a^3b + 7350a^2b^2 - 6860ab^3 + 2401b^4$.

15. $\left(1 + \frac{a}{2}\right)^5$. *Ans.* $1 + \frac{5a}{2} + \frac{5a^2}{2} + \frac{5a^3}{4} + \frac{5a^4}{16} + \frac{a^5}{32}$.

16. $\left(\frac{1}{2}a - 2b\right)^7$.

$$Ans.\ \frac{a^7}{128} - \frac{7}{32}a^6b + \frac{21}{8}a^5b^2 - \frac{35}{2}a^4b^3 + 70a^3b^4$$
$$- 168a^2b^5 + 224ab^6 - 128b^7.$$

17. $(a^2 - ax)^{10}$.

$$Ans.\ a^{20} - 10a^{19}x + 45a^{18}x^2 - 120a^{17}x^3 + 210a^{16}x^4$$
$$- 252a^{15}x^5 + 210a^{14}x^6 - 120a^{13}x^7 + 45a^{12}x^8$$
$$- 10a^{11}x^9 + a^{10}x^{10}.$$

18. $(x^{-2} + a^{-3})^5$.

Ans. $x^{-10} + 5a^{-3}x^{-8} + 10a^{-6}x^{-6} + 10a^{-9}x^{-4} + 5a^{-12}x^{-2} + a^{-15}$.

209. To find any power of a polynomial by means of the binomial theorem.

EXERCISES.

1. Expand $(a + b + c - d)^3$.

Put $x = a + b$, and $y = c - d$;

then $(a+b+c-d)^3 = (x+y)^3 = x^3+3x^2y+3xy^2+y^3$.

Restoring the values of x and y,

$$\begin{aligned}(a+b+c-d)^3 &= (a + b)^3 + 3\,(a + b)^2\,(c - d) \\ &\quad + 3\,(a + b)\,(c - d)^2 + (c - d)^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c \\ &\quad - 3a^2d - 6abd - 3b^2d + 3ac^2 - 6acd \\ &\quad + 3ad^2 + 3bc^2 - 6bcd + 3bd^2 + c^3 - 3c^2d \\ &\quad + 3cd^2 - d^3.\end{aligned}$$

Expand:

2. $(a + b + c)^3$.

Ans. $a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3ac^2$
$+ 3bc^2 + 6abc$.

3. $(1 + x - x^2)^3$. *Ans.* $1 + 3x - 5x^3 + 3x^5 - x^6$.

4. $(ax + by + cz)^3$.

Ans. $a^3x^3 + b^3y^3 + c^3z^3 + 3a^2bx^2y + 3a^2cx^2z + 3ab^2xy^2$
$+ 3b^2cy^2z + 3ac^2xz^2 + 3bc^2yz^2 + 6abcxyz$.

5. $[(a + b)\,x - (c + d)\,y]^3$.

Ans. $(a+b)^3x^3 - 3\,(a+b)^2x^2\,(c+d)\,y + 3\,(a+b)\,x\,(c+d)^2y^2$
$- (c + d)^3\,y^3$.

6. $(1 + x + x^2)^4$.

Ans. $(1+x)^4 + 4\,(1+x)^3x^2 + 6\,(1+x)^2x^4 + 4\,(1+x)\,x^6 + x^8$.

Simplify:

7. $(a + b + c)^3 - 3\,(a+b+c)^2c + 3\,(a+b+c)\,c^2 - c^3$.

Ans. $a^3 + 3a^2b + 3ab^2 + b^3$.

8. $(a-b)^3 + 3\,(a-b)^2(b-c) + 3\,(a-b)\,(b-c)^2 + (b-c)^3$.

Ans. $a^3 - 3a^2c + 3ac^2 - c^3$.

9. $(1 + x + 3x^2 + 3x^3)^3 + (1 - x + 3x^2 - 3x^3)^3$.

Ans. $2\,(1 + 12x^2 + 54x^4 + 108x^6 + 81x^8)$.

10. $[(x + y)^3 - (x^3 + y^3)]^3 - 27x^3y^3\,(x^3 + y^3)$.

Ans. $81x^4y^4\,(x + y)$.

11. $(2 + 3x + 4x^2)^2 + (2 - 3x + 4x^2)^2$.

Ans. $2\,(4 + 25x^2 + 16x^4)$.

12. $(2 + 3x + 4x^2)^3 - (2 - 3x + 4x^2)^3$.

Ans. $2\,(36x + 171x^3 + 144x^5)$.

13. $(1 - x)^3\,(1 + x + x^2)^3$. *Ans.* $1 - 3x^3 + 3x^6 - x^9$.

14. $(1 - x + x^2)^3\,(1 + x + x^2)^3$.

Ans. $1 + 3x^2 + 6x^4 + 7x^6 + 6x^8 + 3x^{10} + x^{12}$.

EVOLUTION.

210. Evolution is the process of finding any root of a quantity.

Evolution is the converse of Involution. In Involution, the root is given, and the *power* is to be found; in Evolution the *power* is given, and the *root* is to be found.

211. If the *index* of the root to be found is odd, the sign of the root will be the same as that of the given quantity **(198, 199)**. Thus, $\sqrt[3]{8} = 2$, and $\sqrt[3]{-8} = -2$.

212. If the *index* of the root is *even*, and the given quantity is *positive*, the roots may be either positive or negative **(198, 199)**. Thus, $\sqrt{9} = +3$ or -3.

213. If the *index* of the root is *even*, and the given quantity is negative, the root cannot be found, for it is impossible to produce a negative result by raising a *negative* quantity to an even power **(199)**.

The indicated even root of a negative quantity is called an *Imaginary Quantity*. Thus, $\sqrt{-9}$ is an imaginary quantity.

214. To find any root of a monomial.

Since Evolution is the converse of Involution, any root of a monomial may be found by reversing the process of Art. **202.** Thus, the square root may be found by extracting the square root of the coefficient, and dividing the exponent of each letter by 2; the cube root may be found by extracting the cube root of the coefficient, and dividing the exponent of each letter by 3; and so on. Thus,

$$\sqrt{9a^4b^4} = \pm 3a^2b^2, \quad \text{and} \quad \sqrt[3]{8a^6b^6} = 2a^2b^2.$$

EXERCISES.

1. Find the square root of $64a^2b^4c^6$. *Ans.* $\pm 8ab^2c^3$.
2. Find the cube root of $27a^3b^6c^9$. *Ans.* $3ab^2c^3$.
3. Find the cube root of $-64a^3b^9c^{12}$. *Ans.* $-4ab^3c^4$.
4. Find the fifth root of $-a^5b^5c^{10}$. *Ans.* $-abc^2$.
5. Has $-a^4b^6c^2$ a square root? Why?
6. Find the fourth root of $16a$. *Ans.* $\pm 2a^{\frac{1}{4}}$.
7. Find the fifth root of $-32x^{10}y^3$. *Ans.* $-2x^2y^{\frac{3}{5}}$.
8. Find the square root of $81a^{-4}b^6$. *Ans.* $\pm 9a^{-2}b^3$.
9. Find the n^{th} root of $a^{3n}b^{2m}$. *Ans.* $a^3b^{\frac{2m}{n}}$.
10. Find the cube root of $-216a^{-3n}c^{-1}$. *Ans.* $-6a^{-n}c^{-\frac{1}{3}}$.
11. Find the fourth root of a^2b^2. *Ans.* $\pm a^{\frac{1}{2}}b^{\frac{1}{2}}$.
12. Find the sixth root of a^2b^4. *Ans.* $\pm a^{\frac{1}{3}}b^{\frac{2}{3}}$.

215. To find the square root of a polynomial.

Since $(a+b+c)^2 = a^2+2ab+b^2+2ac+2bc+c^2$,

it follows that

$$\sqrt{a^2+2ab+b^2+2ac+2bc+c^2} = a+b+c.$$

Now a rule for finding the square root of a polynomial may be deduced by observing how $a+b+c$ may be derived from its square.

$$\begin{array}{r|l|l}
 & a^2+2ab+b^2+2ac+2bc+c^2 & a+b+c. \\
 & a^2 & \\ \hline
2a+b & 2ab+b^2+2ac+2bc+c^2 & \\
 & 2ab+b^2 & \\ \hline
2a+2b+c & 2ac+2bc+c^2 & \\
 & 2ac+2bc+c^2 & \\ \hline
\end{array}$$

Arrange the power according to the descending powers of a; then the first term, a, of the root is the square root of the first term, a^2, of the power. Subtract the square of a from the power; then the second term, b, of the root may be derived from the first term, $2ab$, of the remainder, by dividing it by $2a$. Add b to $2a$, multiply the sum by b, and subtract the product from the remainder. The third term, c, of the root may now be derived from the first term, $2ac$, of the second remainder, by dividing it by $2a$. Add c to $2(a+b)$, multiply the sum by c, and subtract the product from the second remainder. The process terminates here, for the remainder is 0. In this operation, $2a$ and $2(a+b)$ are called *partial divisors*, and $2a+b$ and $2a+2b+c$ are called *complete divisors.*

RULE.

1. *Having arranged the given polynomial according to the powers of one of its letters, extract the square root of its first term. The result will be the first term of the root. Subtract the square of this term from the given polynomial.*

2. *Divide the first term of the remainder by twice the first term of the root. The quotient will be the second term of the root. Annex this term to the first term of the root and to the partial divisor; then multiply the complete divisor by the second term of the root, and subtract the product from the first remainder.*

3. *Multiply the part of the root already found by 2. The product will be the second partial divisor. Divide the first term of the second remainder by the first term of the second divisor. The quotient will be the third term of the root. Annex this term to that part of the root already found and to the partial divisor; then multiply the complete divisor by the third term of the root, and subtract the product from the second remainder.*

4. *If the root contains additional terms, find them in the same manner.*

EXERCISES.

Find the square root of

1. $25a^2 + 20ab + 4b^2$. *Ans.* $5a + 2b$.
2. $x^2 + 8x + 16$. *Ans.* $x + 4$.
3. $x^4 + 20x^2 + 100$. *Ans.* $x^2 + 10$.
4. $a^6 + 12a^3 + 36$. *Ans.* $a^3 + 6$.
5. $9a^2 + 12a + 4$. *Ans.* $3a + 2$.
6. $25x^4 - 70x^2 + 49$. *Ans.* $5x^2 - 7$.
7. $36a^6 - 60a^3 + 25$. *Ans.* $6a^3 - 5$.
8. $x^4 - 4x^3 + 12x^2 - 16x + 16$. *Ans.* $x^2 - 2x + 4$.
9. $1 + 4x + 8x^2 + 8x^3 + 4x^4$. *Ans.* $1 + 2x + 2x^2$.
10. $x^4 - 4x^3 + 2x^2 + 4x + 1$. *Ans.* $x^2 - 2x - 1$.
11. $x^4 + 2x^3 + 3x^2 + 2x + 1$. *Ans.* $x^2 + x + 1$.
12. $16 - 24b + 9b^2 + 8c - 6bc + c^2$.
    Ans. $4 - 3b + c$.
13. $1 + 4x + 10x^2 + 12x^3 + 9x^4$. *Ans.* $1 + 2x + 3x^2$.
14. $9x^4 + 12x^3 + 22x^2 + 12x + 9$.
    Ans. $3x^2 + 2x + 3$.
15. $9a^2 + 12ab + 4b^2 + 6ac + 4bc + c^2$.
    Ans. $3a + 2b + c$.
16. $x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$.
    Ans. $x^2 - 4xy + 4y^2$.
17. $4a^4 - 12a^3 + 25a^2 - 24a + 16$.
    Ans. $2a^2 - 3a + 4$.
18. $16x^4 - 16abx^2 + 16b^2x^2 + 4a^2b^2 - 8ab^3 + 4b^4$.
    Ans. $4x^2 - 2ab + 2b^2$.
19. $x^6 - 4x^5 + 10x^4 - 20x^3 + 25x^2 - 24x + 16$.
    Ans. $x^3 - 2x^2 + 3x - 4$.
20. $9a^2 - 6ab + 30ac + 6ad + b^2 - 10bc - 2bd + 25c^2 + 10cd + d^2$. *Ans.* $3a - b + 5c + d$.

21. $x^6 - 4x^5y + 8x^4y^2 - 10x^3y^3 + 8x^2y^4 - 4xy^5 + y^6$.
Ans. $x^3 - 2x^2y + 2xy^2 - y^3$.

22. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.
Ans. $1 - 3x + 3x^2 - x^3$.

23. $4 - 12a + 5a^2 + 14a^3 - 11a^4 - 4a^5 + 4a^6$.
Ans. $2 - 3a - a^2 + 2a^3$.

24. $1 - 4x + 10x^2 - 20x^3 + 25x^4 - 24x^5 + 16x^6$.
Ans. $1 - 2x + 3x^2 - 4x^3$.

25. $p^2 + 2pqx + (2pr + q^2)x^2 + 2(ps + qr)x^3 + (2qs + r^2)x^4 + 2rsx^5 + s^2x^6$.
Ans. $p + qx + rx^2 + sx^3$.

216. To find the cube root of a polynomial.

Since $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,

it follows that

$$\sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3} = a + b.$$

Hence, a rule for finding the cube root of a polynomial may be deduced by observing how $a + b$ may be derived from its cube.

$$\begin{array}{r|l|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 & a + b. \\
 & a^3 & \\
\hline
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 & \\
 & 3a^2b + 3ab^2 + b^3 & \\
\hline
\end{array}$$

Arrange the power according to the descending powers of a; then the first term, a, of the root is the cube root of the first term, a^3, of the power. Subtract the cube of a from the power; then the second term, b, of the root may be derived from the first term, $3a^2b$, of the remainder by dividing it by $3a^2$. Add $3ab + b^2$ to $3a^2$, multiply the sum by b, and subtract the product from the remainder. The process terminates here, for the second remainder is 0. In this operation, $3a^2$ is called the *partial divisor*, and $3a^2 + 3ab + b^2$, the *complete divisor*.

If it be required to extract the cube root of $a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$, which is the cube of $a + b + c$, find the first and second terms, a and b, in the manner just explained; then divide the first term, $3a^2c$, of the second remainder by the first term of $3(a+b)^2$. The quotient, c, will be the third term of the root. Then add $3(a+b)c + c^2$ to $3(a+b)^2$, multiply the sum by c, and subtract the product from the second remainder.

RULE.

1. *Having arranged the given polynomial according to the powers of one of its letters, extract the cube root of its first term. The result will be the first term of the root. Subtract the cube of this term from the given polynomial.*

2. *Divide the first term of the remainder by three times the square of the first term of the root. The quotient will be the second term of the root. Annex this term to the first term of the root, and add to the partial divisor three times the product of the first and second terms of the root and the square of the second term; then multiply the complete divisor by the second term of the root, and subtract the product from the first remainder.*

3. *Multiply the square of the part of the root already found by 3. The product will be the second partial divisor. Divide the first term of the second remainder by the first term of the second divisor. The quotient will be the third term of the root. Annex this term to that part of the root already found, and to the partial divisor add three times the product of the sum of the first and second terms of the root by the third, and the square of the third term; then multiply the complete divisor by the third term of the root, and subtract the product from the second remainder.*

4. *If the root contains additional terms, find them in the same manner.*

EXERCISES.

Find the cube root of

1. $a^3 + 12a^2 + 48a + 64.$ *Ans.* $a + 4.$
2. $a^3 - 3a^2x + 3ax^2 - x^3.$ *Ans.* $a - x.$
3. $8x^3 + 12x^2 + 6x + 1.$ *Ans.* $2x + 1.$
4. $8x^3 - 36ax^2 + 54a^2x - 27a^3.$ *Ans.* $2x - 3a.$
5. $27x^3 - 135x^2 + 225x - 125.$ *Ans.* $3x - 5.$
6. $a^3x^6 - 12a^2x^4 + 48ax^2 - 64.$ *Ans.* $ax^2 - 4.$
7. $8a^6 - 36a^4b^2 + 54a^2b^4 - 27b^6.$ *Ans.* $2a^2 - 3b^2.$
8. $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1.$ *Ans.* $a^2 - 2a + 1.$
9. $27a^6 - 108a^5x + 90a^4x^2 + 80a^3x^3 - 60a^2x^4 - 48ax^5 - 8x^6.$ *Ans.* $3a^2 - 4ax - 2x^2.$
10. $a^3 + b^3 - c^3 - 6abc + 3a^2b + 3ab^2 - 3a^2c - 3b^2c + 3ac^2 + 3bc^2.$ *Ans.* $a + b - c.$
11. $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1.$ *Ans.* $a^2 + 2a + 1.$
12. $x^6 - 12x^5 + 54x^4 - 112x^3 + 108x^2 - 48x + 8.$ *Ans.* $x^2 - 4x + 2.$

217. To find the higher roots of quantities.

If the index of the required root contains no prime factor greater than 3, the root may be found by methods already explained, for

$$\sqrt[m]{\sqrt[n]{(x+a)^{mn}}} = \sqrt[m]{(x+a)^m} = x + a,$$

and

$$\sqrt[mn]{(x+a)^{mn}} = x + a;$$

$$\therefore \quad \sqrt[mn]{(x+a)^{mn}} = \sqrt[m]{\sqrt[n]{(x+a)^{mn}}};$$ that is,

The mn^{th} root of a quantity is equal to the m^{th} root of the n^{th} root of that quantity. Thus,

$$\sqrt[4]{81} = \sqrt{\sqrt{81}} = \sqrt{9} = 3,$$

$$\sqrt[6]{64} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2,$$

and $$\sqrt[8]{256} = \sqrt{\sqrt{\sqrt{256}}} = \sqrt{\sqrt{16}} = \sqrt{4} = 2.$$

Hence, the fourth root of a quantity may be found by extracting the square root of the square root of that quantity; the sixth root, by extracting the square root, then the cube root of the result; and so on.

EXERCISES.

1. Find the fourth root of $6a^2b^2 + a^4 - 4a^3b - 4ab^3 + b^4$.
Ans. $a - b$.

2. Find the sixth root of $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.
Ans. $a + b$.

3. Find the eighth root of $a^8 - 16a^7 + 112a^6 - 448a^5 + 1120a^4 - 1792a^3 + 1792a^2 - 1024a + 256$.
Ans. $a - 2$.

4. Find the ninth root of $a^9 - 9a^8c + 36a^7c^2 - 84a^6c^3 + 126a^5c^4 - 126a^4c^5 + 84a^3c^6 - 36a^2c^7 + 9ac^8 - c^9$
Ans. $a - c$.

218. To find any root of a fraction.

The roots of fractions may be found by means of the following principle:

The n^{th} root of a fraction is a fraction whose numerator is the n^{th} root of the given numerator, and whose denominator is the n^{th} root of the given denominator.

Thus, $$\sqrt{\frac{9}{16}} = \frac{3}{4}, \quad \text{for} \quad \left(\frac{3}{4}\right)^2 = \frac{9}{16},$$

and $$\sqrt[n]{\frac{a^n}{b^n}} = \frac{a}{b}, \quad \text{for} \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

EXERCISES.

1. Find the square root of $\frac{x^2-2ax+a^2}{x^2+2ax+a^2}$. *Ans.* $\frac{x-a}{x+a}$.

2. Find the cube root of $\frac{8x^3-12x^2+6x-1}{x^6+6x^5-40x^3+96x-64}$.

Ans. $\frac{2x-1}{x^2+2x-4}$.

3. Find the fourth root of

$$\frac{6a^2b^2+a^4-4a^3b-4ab^3+b^4}{16a^4-128a^3b+384a^2b^2-512ab^3+256b^4}.$$

Ans. $\frac{a-b}{2a-4b}$.

4. Find the sixth root of $\frac{(a^2-2ab+b^2)^3}{(a^3+3a^2b+3ab^2+b^3)^2}$.

Ans. $\frac{a-b}{a+b}$.

5. Find the seventh root of $-\frac{a^7b^{14}c^5d^3}{x^2y^3z^4}$. *Ans.* $-\frac{ab^2c^{\frac{5}{7}}d^{\frac{3}{7}}}{x^{\frac{2}{7}}y^{\frac{3}{7}}z^{\frac{4}{7}}}$.

FRACTIONAL EXPONENTS.

219. By Art. 214,

$$\sqrt{a}=a^{\frac{1}{2}}, \qquad \sqrt{a^3}=a^{\frac{3}{2}},$$

$$\sqrt[3]{a}=a^{\frac{1}{3}}, \qquad \sqrt[3]{a^2}=a^{\frac{2}{3}}, \text{ etc.} \quad \text{Hence,}$$

The numerator of a positive fractional exponent indicates the power to which the quantity affected by it is to be raised, and the denominator indicates the root which is to be extracted.

EXERCISES.

Express by means of fractional exponents:

1. $\sqrt{x^5}$. *Ans.* $x^{\frac{5}{2}}$.
2. $\sqrt[3]{a^5b^2}$. *Ans.* $a^{\frac{5}{3}}b^{\frac{2}{3}}$.
3. $\sqrt[4]{x^5y^3}$. *Ans.* $x^{\frac{5}{4}}y^{\frac{3}{4}}$.
4. $\sqrt[8]{a^3b^5}$. *Ans.* $a^{\frac{3}{8}}b^{\frac{5}{8}}$.
5. $\dfrac{xy}{\sqrt{ab}}$. *Ans.* $\dfrac{xy}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$.
6. $\dfrac{\sqrt[n]{x^m}}{\sqrt[m]{a^n}}$. *Ans.* $\dfrac{x^{\frac{m}{n}}}{a^{\frac{n}{m}}}$.

220. The principles relating to negative integral exponents (**141**) are applicable to negative fractional exponents.

$$a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}, \quad \text{for} \quad a^{-\frac{2}{3}} = \sqrt[3]{a^{-2}} = \sqrt[3]{\frac{1}{a^2}} = \frac{1}{a^{\frac{2}{3}}}.$$

EXERCISES.

Express by means of positive exponents:

1. $2a^{-\frac{1}{2}}b^{-\frac{1}{3}}$. *Ans.* $\dfrac{2}{a^{\frac{1}{2}}b^{\frac{1}{3}}}$.
2. $\dfrac{3a}{x^{-\frac{1}{3}}y^{-\frac{3}{4}}}$. *Ans.* $3ax^{\frac{1}{3}}y^{\frac{3}{4}}$.
3. $\dfrac{a^{-\frac{1}{3}}}{b^{-\frac{2}{3}}c^{-\frac{1}{5}}}$. *Ans.* $\dfrac{b^{\frac{2}{3}}c^{\frac{1}{5}}}{a^{\frac{1}{3}}}$.
4. $\dfrac{1}{a^{-\frac{3}{8}}b^{-\frac{5}{8}}}$. *Ans.* $a^{\frac{3}{8}}b^{\frac{5}{8}}$.
5. $\dfrac{5(x+y)^{-\frac{1}{2}}}{6(x-y)^{-\frac{2}{3}}}$. *Ans.* $\dfrac{5(x-y)^{\frac{2}{3}}}{6(x+y)^{\frac{1}{2}}}$.

CHAPTER X.

RADICAL QUANTITIES.

221. A **Radical Quantity** is an expression which contains the radical sign. Thus, $2\sqrt{3}$, $3\sqrt[3]{4}$, $2 \times 5^{\frac{1}{2}}$, $x + \sqrt[3]{a}$ are radical quantities.

Radical quantities are often called, simply, *radicals.*

222. A **Simple Radical** is a monomial containing a single radical sign. Thus, $2\sqrt{3}$, $\sqrt[3]{5}$, $a\sqrt{x + c}$ are simple radicals.

223. The **Radical Factor** of a simple radical is the indicated root, and the **Coefficient** of the radical factor is the other factor of the expression. Thus, in the expression $2\sqrt{3}$, the radical factor is $\sqrt{3}$, and 2 is its coefficient.

224. The **Degree** of a simple radical is denoted by the index of the indicated root. Thus, $2\sqrt{3}$, $2 \times 5^{\frac{3}{2}}$ are of the second degree, and $2\sqrt[3]{3}$, $2 \times 5^{\frac{2}{3}}$ are of the third degree.

225. **Similar Simple Radicals** are those which have a common radical factor. Thus, $5\sqrt[3]{3}$ and $6\sqrt[3]{3}$ are similar.

226. A simple radical is in its **Simplest Form**, if the quantity under the radical sign is entire, and contains no factor which is a perfect power corresponding to the degree of the radical.

227. A **Polynomial Radical** is a radical having two or more terms. Thus, $2 + \sqrt{3}$ and $2\sqrt{3} + 3\sqrt[3]{5} - 4\sqrt[4]{6}$ are polynomial radicals.

228. A **Complex Radical** is a radical in which one radical sign includes at least one other. Thus,

$$\sqrt{\sqrt{8}} \text{ and } \sqrt{9 + 3\sqrt{\sqrt{5}}} \text{ are complex radicals.}$$

229. A **Rational Quantity** is a quantity which can be exactly expressed without the use of the radical sign. Thus, 3, -4, 5^2, and $\sqrt{25}$ are rational.

Any rational quantity can be expressed under the form of a radical. Thus,

$$5 = \sqrt{25}, \quad -4 = -\sqrt{16}, \quad \text{and} \quad x + a = \sqrt{(x+a)^2}.$$

230. An **Irrational Quantity** is a quantity which cannot be exactly expressed without the use of the radical sign. Thus, $3\sqrt{5}$, $2\sqrt[3]{9}$, and $5\sqrt[4]{2}$ are irrational.

Irrational quantities are often called *surd quantities*, or, simply, *surds*.

231. An **Imaginary Quantity** is an expression which, in its simplest form, contains an indicated *even root* of a negative quantity. Thus, $3\sqrt{-1}$, $5\sqrt[4]{-16}$, and $2\sqrt[6]{-64}$ are imaginary.

232. Imaginary quantities are classified in the same way as other surds. Thus,

$$3\sqrt{-1} \text{ is simple and of the second degree,}$$

$$2 + 3\sqrt{-1} + 5\sqrt[4]{-16} \text{ is a polynomial,}$$

and $$\sqrt{-1 + \sqrt{-4}} \text{ is complex.}$$

233. A **Real Quantity** is an expression which, in its simplest form, does not contain an indicated *even root* of a negative quantity. Thus, 8, -5, $\sqrt{8}$, and $\sqrt[3]{-8}$ are real.

REDUCTION OF SIMPLE RADICALS.

234. TH. *The product of the square roots of two or more quantities is equal to the square root of their product; the product of their cube roots is equal to the cube root of their product; and so on.* Thus,

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab};$$

for
$$\begin{aligned}(\sqrt{a} \times \sqrt{b})^2 &= (\sqrt{a} \times \sqrt{b})(\sqrt{a} \times \sqrt{b}) \\ &= \sqrt{a} \times \sqrt{a} \times \sqrt{b} \times \sqrt{b} \\ &= ab;\end{aligned}$$

that is, the square of $\sqrt{a} \times \sqrt{b}$ is equal to ab;

$$\therefore \quad \sqrt{a} \times \sqrt{b} = \sqrt{ab}.$$

In like manner, it may be shown that

$$\sqrt[3]{a} \times \sqrt[3]{b} = \sqrt[3]{ab};$$

$$\sqrt[4]{a} \times \sqrt[4]{b} = \sqrt[4]{ab}; \quad \text{and so on.}$$

235. TH. *The quotient of the square roots of two quantities is equal to the square root of their quotient; the quotient of their cube roots is equal to the cube root of their quotient; and so on.* Thus,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}};$$

for
$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a} \times \sqrt{a}}{\sqrt{b} \times \sqrt{b}} = \frac{a}{b};$$

that is, the square of $\frac{\sqrt{a}}{\sqrt{b}}$ is equal to $\frac{a}{b}$;

$$\therefore \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

In like manner, it may be shown that

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \sqrt[3]{\frac{a}{b}};$$

$$\frac{\sqrt[4]{a}}{\sqrt[4]{b}} = \sqrt[4]{\frac{a}{b}}; \quad \text{and so on.}$$

236. To reduce a rational quantity to a radical of any degree.

$$2 = \sqrt{4} = \sqrt[3]{8} = \sqrt[4]{16} = \sqrt[n]{2^n};$$

$$a = \sqrt{a^2} = \sqrt[3]{a^3} = \sqrt[4]{a^4} = \sqrt[n]{a^n}.$$

RULE.

Raise the given quantity to the power corresponding to the degree of the radical, and put the result under the radical sign of the given degree.

EXERCISES.

1. Reduce a^2 to a radical of the second degree.
Ans. $\sqrt{a^4}$.

2. Reduce $a-x$ to a radical of the third degree.
Ans. $\sqrt[3]{(a-x)^3}$.

3. Reduce -5 to a radical of the second degree.
Ans. $-\sqrt{25}$.

4. Reduce -7 to a radical of the third degree.
Ans. $\sqrt[3]{-343}$.

237. To put the coefficient of the radical factor under the radical sign.

$$2\sqrt{3} = \sqrt{4} \times \sqrt{3} = \sqrt{12} \quad \textbf{(236, 234)};$$

$$a\sqrt[3]{x} = \sqrt[3]{a^3} \times \sqrt[3]{x} = \sqrt[3]{a^3x}.$$

RULE.

Raise the coefficient to the power corresponding to the degree of the radical, and write the result as a factor under the radical sign.

Cor.—In the same way any *factor* of the coefficient may be put under the radical sign.

Thus, $6\sqrt{5} = 3 \times 2\sqrt{5} = 3\sqrt{5 \times 4} = 3\sqrt{20}$.

EXERCISES.

Put the coefficient in each of the following radicals under the radical sign:

1. $2\sqrt{3}$. *Ans.* $\sqrt{12}$.
2. $3\sqrt[3]{2}$. *Ans.* $\sqrt[3]{54}$.
3. $7\sqrt{ab}$. *Ans.* $\sqrt{49ab}$.
4. $3\sqrt{\frac{2}{3}}$. *Ans.* $\sqrt{6}$.
5. $x\sqrt{\frac{a}{x}}$. *Ans.* $\sqrt{ax}$.
6. $\frac{1}{3}\sqrt[3]{12}$. *Ans.* $\sqrt[3]{\frac{4}{9}}$.
7. $(a-x)\sqrt{\frac{c}{a^2-2ax+x^2}}$. *Ans.* $\sqrt{c}$.
8. $\frac{a}{x}\left(\frac{x}{a}+\frac{a}{x}\right)^{\frac{1}{2}}$. *Ans.* $\left(\frac{a}{x}+\frac{a^3}{x^3}\right)^{\frac{1}{2}}$.

238. To move a factor of the quantity under the radical sign to the coefficient.

The reduction is performed by reversing the process of Art. **237**. Thus,

$$3\sqrt[3]{16} = 3\sqrt[3]{8 \times 2} = 3\sqrt[3]{8} \times \sqrt[3]{2} \ (\mathbf{234})$$
$$= 3 \times 2\sqrt[3]{2} = 6\sqrt[3]{2}.$$

RULE.

Extract the indicated root of the factor to be moved, and write the result as a factor in the coefficient.

EXERCISES.

1. In $\sqrt{20}$ move $\sqrt{4}$ to the coefficient. *Ans.* $2\sqrt{5}$.

2. In $3\sqrt[3]{24}$ move $\sqrt[3]{8}$ to the coefficient. *Ans.* $6\sqrt[3]{3}$.

3. In $4\sqrt{75}$ move $\sqrt{25}$ to the coefficient. *Ans.* $20\sqrt{3}$.

4. In $(a-x)\sqrt{(a+x)^2 m}$ move $\sqrt{(a+x)^2}$ to the coefficient. *Ans.* $(a^2-x^2)\sqrt{m}$.

5. In $\sqrt[3]{\frac{3}{27}}$ move $\sqrt[3]{\frac{1}{27}}$ to the coefficient. *Ans.* $\frac{1}{3}\sqrt[3]{3}$.

6. In $(a-b)\sqrt{(a+b)^3 cd}$ move $\sqrt{(a+b)^2}$ to the coefficient. *Ans.* $(a^2-b^2)\sqrt{(a+b)cd}$.

7. In $3\sqrt[3]{\frac{75}{125}}$ move $\sqrt[3]{\frac{1}{125}}$ to the coefficient.
Ans. $\frac{3}{5}\sqrt[3]{75}$.

8. Show that $\sqrt[3]{\frac{12}{32}} = \frac{1}{2}\sqrt[3]{3}$.

239. To reduce the indicated root of a fraction to a radical in which the quantity under the radical sign shall be entire.

$$2\sqrt{\frac{5}{9}} = 2\sqrt{\frac{1}{9}\times 5} = \frac{2}{3}\sqrt{5} \ \textbf{(238)};$$

$$3\sqrt{\frac{3}{5}} = 3\sqrt{\frac{3\times 5}{5\times 5}} = 3\sqrt{\frac{15}{25}} = 3\sqrt{\frac{1}{25}\times 15} = \frac{3}{5}\sqrt{15};$$

$$\sqrt[3]{\frac{24}{32}} = \sqrt[3]{\frac{24\div 4}{32\div 4}} = \sqrt[3]{\frac{6}{8}} = \sqrt[3]{\frac{1}{8}\times 6} = \frac{1}{2}\sqrt[3]{6}.$$

RULES.

I. *If the denominator of the fraction is a perfect power of the degree of the radical, extract the indicated root of the denominator and put this root as a divisor of the coefficient.*

II. *If the denominator of the fraction is not a perfect power of the degree of the radical, multiply or divide both terms of the fraction by such a quantity as will make the denominator a perfect power of the required degree; then proceed as in I.*

EXERCISES.

Reduce each of the following radicals to another in which the quantity under the radical sign shall be entire:

1. $5\sqrt{\frac{6}{9}}$. *Ans.* $\frac{5}{3}\sqrt{6}$.

2. $2\sqrt{\frac{2}{3}}$. *Ans.* $\frac{2}{3}\sqrt{6}$.

3. $\frac{3}{5}\sqrt{1\frac{2}{3}}$. *Ans.* $\frac{1}{5}\sqrt{15}$.

4. $4\sqrt{\frac{3}{4}}$. *Ans.* $2\sqrt{3}$.

5. $\frac{5}{2}\left(\frac{24}{5}\right)^{\frac{1}{3}}$. *Ans.* $\frac{1}{2}(600)^{\frac{1}{3}}$.

6. $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}}$. *Ans.* $\frac{1}{3}\sqrt[3]{18a}$.

7. $\sqrt[3]{\frac{12}{32}}$. *Ans.* $\frac{1}{2}\sqrt[3]{3}$.

8. $\sqrt[3]{\frac{3}{5}}$. *Ans.* $\frac{1}{5}\sqrt[3]{75}$.

9. $\sqrt[n]{\frac{a}{b}}$. Ans. $\frac{1}{b}\sqrt[n]{ab^{n-1}}$.

10. $b(a+c)\sqrt[n]{\frac{a-c}{a+c}}$. Ans. $b\sqrt[n]{(a-c)(a+c)^{n-1}}$.

240. To reduce a simple radical to its simplest form.

1. Let it be required to reduce $5\sqrt{8}$ to its simplest form.

$$5\sqrt{8} = 5\sqrt{4 \times 2} = 10\sqrt{2}.$$

The greatest perfect square that is a factor of 8 is 4. Moving the square root of 4 to the coefficient, the result is $10\sqrt{2}$.

2. Let it be required to reduce $5\sqrt[3]{48}$ to its simplest form.

$$5\sqrt[3]{48} = 5\sqrt[3]{8 \times 6} = 10\sqrt[3]{6}.$$

The greatest perfect cube that is a factor of 48 is 8. Moving the cube root of 8 to the coefficient, the result is $10\sqrt[3]{6}$.

3. Let it be required to reduce $2\sqrt{\frac{8}{3}}$ to its simplest form.

$$2\sqrt{\frac{8}{3}} = 2\sqrt{\frac{24}{9}} = \frac{2}{3}\sqrt{24} = \frac{2}{3}\sqrt{4 \times 6} = \frac{4}{3}\sqrt{6}.$$

RULES.

I. *If the quantity under the radical sign is entire, resolve it into two factors, one of which is the greatest perfect power of the degree of the radical; then extract the indicated root of this factor, and put the root as a factor of the coefficient* (**238**).

II. *If the radical contains the indicated root of a fraction, reduce it to another in which the quantity under the radical sign shall be entire* (**239**), *then proceed as in I.*

EXERCISES.

Reduce each of the following radicals to its simplest form:

1. $3\sqrt{75}$. *Ans.* $15\sqrt{3}$.

2. $2\sqrt{27a^3}$. *Ans.* $6a\sqrt{3a}$.

3. $3\sqrt{192a^5b^8c^7}$. *Ans.* $24a^2b^4c^3\sqrt{3ac}$.

4. $2\sqrt[3]{108a^3b}$. *Ans.* $6a\sqrt[3]{4b}$.

5. $5\sqrt{\frac{24}{9}}$. *Ans.* $\frac{10}{3}\sqrt{6}$.

6. $2\sqrt{2\frac{2}{3}}$. *Ans.* $\frac{4}{3}\sqrt{6}$.

7. $\frac{3}{5}\sqrt{6\frac{2}{3}}$. *Ans.* $\frac{2}{5}\sqrt{15}$.

8. $\frac{5}{2}\left(\frac{24}{5}\right)^{\frac{1}{3}}$. *Ans.* $(75)^{\frac{1}{3}}$.

9. $\frac{2a}{3}\left(\frac{9}{4a^2}\right)^{\frac{1}{3}}$. *Ans.* $\frac{1}{3}(18a)^{\frac{1}{3}}$.

10. $\frac{a}{b}\sqrt{\frac{c^2}{d}}$. *Ans.* $\frac{ac}{bd}\sqrt{d}$.

11. $(x+a)\sqrt{\frac{x-a}{x+a}}$. *Ans.* $\sqrt{x^2-a^2}$.

12. $\sqrt[4]{\frac{a-x}{a+x}}$. *Ans.* $\frac{1}{a+x}\sqrt[4]{(a-x)(a+x)^3}$.

241. To reduce simple radicals having unequal indices to others having equal indices.

1. Let it be required to reduce $\sqrt{2}$ and $\sqrt[3]{5}$ to equivalent radicals having equal indices.

$$\sqrt{2} = (2)^{\frac{1}{2}} = (2)^{\frac{3}{6}} = \sqrt[6]{2^3} \ \textbf{(219)};$$
$$\sqrt[3]{5} = (5)^{\frac{1}{3}} = (5)^{\frac{2}{6}} = \sqrt[6]{5^2}.$$

2. Let it be required to reduce $\sqrt[3]{a}$ and $\sqrt[4]{b}$ to equivalent radicals having equal indices.

$$\sqrt[3]{a} = a^{\frac{1}{3}} = a^{\frac{4}{12}} = \sqrt[12]{a^4};$$
$$\sqrt[4]{b} = b^{\frac{1}{4}} = b^{\frac{3}{12}} = \sqrt[12]{b^3}.$$

RULE.

Express the indicated roots by means of fractional exponents; then reduce the expressions thus obtained to others in which the fractional exponents shall have equal denominators.

COR. 1. If equal factors be introduced into the index of the root and the exponent of the quantity under the radical sign, the value of the radical will not be changed. Thus, $\sqrt[3]{a^2} = \sqrt[6]{a^4} = \sqrt[9]{a^6}$.

COR. 2. If equal factors be canceled in the index of the root and the exponent of the quantity under the radical sign, the value of the radical will not be changed. Thus, $\sqrt[9]{a^6} = \sqrt[3]{a^2}$.

EXERCISES.

1. Reduce $2\sqrt{2}$ and $3\sqrt[3]{3}$ to radicals having equal indices. *Ans.* $2\sqrt[6]{8}$, $3\sqrt[6]{9}$.

2. Reduce $\sqrt{2}$, $2\sqrt[3]{3}$, $3\sqrt[4]{4}$ to radicals having equal indices. *Ans.* $\sqrt[6]{8}$, $2\sqrt[6]{9}$, $3\sqrt[6]{8}$.

3. Reduce $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[5]{c}$ to radicals having equal indices. *Ans.* $\sqrt[30]{a^{15}}$, $\sqrt[30]{b^{10}}$, $\sqrt[30]{c^6}$.

4. Reduce $\sqrt[10]{a^5}$, $\sqrt[6]{b^3}$, $\sqrt[8]{c^4}$ to radicals having equal indices. *Ans.* $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$.

5. Reduce $(x+a)^{\frac{2}{3}}$, $(x-a)^{\frac{3}{4}}$, $(m+n)^{\frac{1}{2}}$ to radicals having equal indices. *Ans.* $(x+a)^{\frac{8}{12}}$, $(x-a)^{\frac{9}{12}}$, $(m+n)^{\frac{6}{12}}$.

ADDITION.

242. To find the sum of simple radicals.

1. Let it be required to find the sum of $2\sqrt{3}$ and $5\sqrt{3}$.

$$2\sqrt{3} + 5\sqrt{3} = (2 + 5)\sqrt{3} = 7\sqrt{3}.$$

2. Let it be required to find the sum of $2\sqrt{12}$ and $5\sqrt{27}$.

$$2\sqrt{12} = 2\sqrt{4 \times 3} = 4\sqrt{3};$$
$$5\sqrt{27} = 5\sqrt{9 \times 3} = 15\sqrt{3};$$
$$\therefore \quad 2\sqrt{12} + 5\sqrt{27} = 4\sqrt{3} + 15\sqrt{3} = 19\sqrt{3}.$$

3. Let it be required to find the sum of $3\sqrt{8}$ and $5\sqrt{27}$.

$$3\sqrt{8} = 3\sqrt{4 \times 2} = 6\sqrt{2};$$
$$5\sqrt{27} = 5\sqrt{9 \times 3} = 15\sqrt{3};$$
$$\therefore \quad 3\sqrt{8} + 5\sqrt{27} = 6\sqrt{2} + 15\sqrt{3}.$$

In this example the radicals cannot be made similar; hence, the addition can only be indicated.

4. Let it be required to find the sum of $2\sqrt{5}$ and $3\sqrt[4]{25}$.

$$3\sqrt[4]{25} = 3\sqrt[4]{5^2} = 3\sqrt{5} \quad (\mathbf{241}, \text{ Cor. } 2);$$
$$\therefore \quad 2\sqrt{5} + 3\sqrt[4]{25} = 2\sqrt{25} + 3\sqrt{5} = 5\sqrt{5}.$$

RULES.

I. *If the given radicals are similar, prefix the sum of the coefficients to the common radical factor.*

II. *If the given radicals are of the same degree, but not similar, reduce them, if possible, to similar radicals* (**225**, **238**); *then proceed as in* I. *If they cannot be so reduced, indicate their sum.*

III. *If the given radicals are of different degrees, reduce them to others of the same degree; then proceed as in* II.

EXERCISES.

Find the sum of

1. $7\sqrt{10}$ and $6\sqrt{10}$. *Ans.* $13\sqrt{10}$.

2. $3\sqrt{27}$, $2\sqrt{12}$, $5\sqrt{75}$. *Ans.* $38\sqrt{3}$.

3. $\sqrt{\frac{1}{2}}$, $\frac{1}{3}\sqrt{2}$, $\frac{1}{3}\sqrt{\frac{1}{2}}$. *Ans.* $\sqrt{2}$.

4. $4\sqrt[3]{375}$ and $3\sqrt[3]{81}$. *Ans.* $29\sqrt[3]{3}$.

5. $\sqrt{a^2x}$ and $\sqrt{b^2x}$. *Ans.* $(a + b)\sqrt{x}$.

6. $a\sqrt{6x^2}$, $b\sqrt{24x^2}$, $c\sqrt{54x^2}$. *Ans.* $x(a + 2b + 3c)\sqrt{6}$.

7. $3\sqrt[8]{16a^4b^4}$ and $5\sqrt[4]{4a^2b^2}$. *Ans.* $8\sqrt{2ab}$.

8. $\sqrt{24}$, $2\sqrt{72}$, $2\sqrt{5}$. *Ans.* $2\sqrt{6} + 12\sqrt{2} + 2\sqrt{5}$.

9. $(2ax^2 - 4ax + 2a)^{\frac{1}{2}}$ and $(2ax^2 + 4ax + 2a)^{\frac{1}{2}}$. *Ans.* $2x(2a)^{\frac{1}{2}}$.

10. $a(b^3)^{\frac{1}{6}}$ and $c(b)^{\frac{1}{2}}$. *Ans.* $(a + c)b^{\frac{1}{2}}$.

SUBTRACTION.

243. To find the difference between two simple radicals.

The rules for finding the difference between two simple radicals may be derived from those for finding their sum by substituting the word *difference* for the word *sum*. Let the student repeat those rules with this modification. The difference may be found also by *adding* the negative of the subtrahend to the minuend (**74**).

EXERCISES.

Subtract

1. $2\sqrt{5}$ from $6\sqrt{5}$. *Ans.* $4\sqrt{5}$.
2. $2\sqrt{12}$ from $3\sqrt{27}$. *Ans.* $5\sqrt{3}$.
3. $\frac{1}{3}\sqrt{2}$ from $\sqrt{\frac{1}{2}}$. *Ans.* $\frac{1}{6}\sqrt{2}$.
4. $3\sqrt[3]{81}$ from $4\sqrt[3]{375}$. *Ans.* $11\sqrt[3]{3}$.
5. $\sqrt{b^2x}$ from $\sqrt{a^2x}$. *Ans.* $(a-b)\sqrt{x}$.
6. $b\sqrt{6x^2}$ from $a\sqrt{24x^2}$. *Ans.* $x(2a-b)\sqrt{6}$.
7. $2\sqrt[8]{16a^4b^4}$ from $7\sqrt[4]{4a^2b^2}$. *Ans.* $5\sqrt{2ab}$.
8. $\sqrt{24}$ from $2\sqrt{72}$. *Ans.* $12\sqrt{2}-2\sqrt{6}$.
9. $(ax^2-2ax+a)^{\frac{1}{2}}$ from $(ax^2+2ax+a)^{\frac{1}{2}}$. *Ans.* $2a^{\frac{1}{2}}$.
10. $cb^{\frac{1}{2}}$ from $a(b^3)^{\frac{1}{6}}$. *Ans.* $(a-c)b^{\frac{1}{2}}$.

MULTIPLICATION.

244. To find the product of simple radicals.

1. Let it be required to multiply $2\sqrt{18}$ by $5\sqrt{48}$.

$$\begin{aligned}2\sqrt{18}\times 5\sqrt{48} &= 2\times 5\sqrt{18}\times\sqrt{48}\quad (77,\ \text{Cor. } 3)\\ &= 10\sqrt{18\times 48}\quad (\mathbf{234})\\ &= 10\sqrt{9\times 16\times 2\times 3} = 120\sqrt{6}.\end{aligned}$$

2. Let it be required to multiply $2\sqrt{a}$ by $5\sqrt[3]{2a}$.

$$\begin{aligned}2\sqrt{a}\times 5\sqrt[3]{2a} &= 2\sqrt[6]{a^3}\times 5\sqrt[6]{(2a)^2}\quad (\mathbf{241},\ \text{Cor. } 1)\\ &= 10\sqrt[6]{4a^5}.\end{aligned}$$

RULES.

I. *If the given radicals are of the same degree, find the product of the radical factors by the principle of Art.* **234**, *and to the result prefix the product of their coefficients. Express the final result in its simplest form.*

II. *If the given radicals are of different degrees, reduce them to others of the same degree; then proceed as in* I.

COR. If the radicals are expressed by means of fractional exponents, their product may be found by the rule of Art. **78.** Thus, $2a^{\frac{1}{2}} \times 3a^{\frac{1}{3}} = 6a^{\frac{1}{2}+\frac{1}{3}} = 6a^{\frac{5}{6}}$.

EXERCISES.

Multiply

1. $2\sqrt{5}$ by $6\sqrt{5}$. *Ans.* 60.
2. $3\sqrt{6}$ by $5\sqrt{9}$. *Ans.* $45\sqrt{6}$.
3. $\frac{4}{3}\sqrt[3]{18}$ by $\frac{1}{2}\sqrt[3]{180}$. *Ans.* $4\sqrt[3]{15}$.
4. $2\sqrt[3]{4}$ by $5\sqrt{3}$. *Ans.* $10\sqrt[6]{432}$.
5. $\sqrt{24a^2b}$ by $\sqrt{12b}$. *Ans.* $12ab\sqrt{2}$.
6. $a\sqrt[3]{ax}$ by $b\sqrt{by}$. *Ans.* $ab\sqrt[6]{a^2b^3x^2y^3}$.
7. $5\sqrt[3]{b}$ by $3\sqrt[4]{a}$. *Ans.* $15\sqrt[12]{a^3b^4}$.
8. $a^{\frac{1}{4}}$ by $a^{\frac{3}{4}}$. *Ans.* a.
9. $2a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{4}{5}}$ by $3a^{\frac{1}{4}}b^{\frac{1}{3}}c^{\frac{1}{5}}d^{\frac{3}{5}}$. *Ans.* $6a^{\frac{3}{4}}bcd^{\frac{3}{5}}$.
10. $m(a+b)^{\frac{2}{3}}$ by $n(a-b)^{\frac{2}{3}}$. *Ans.* $mn(a^2-b^2)^{\frac{2}{3}}$.

245. To find the product of quantities, when one or each of them is a polynomial radical.

The product of quantities, when one or each of them is a polynomial radical, may be found by combining the rules of Art. **244** with those of Articles **79** and **80**. If the radicals are expressed by means of fractional exponents, the rules of Articles **79** and **80** are sufficient.

EXERCISES.

1. Multiply $3 + \sqrt{5}$
   by $\sqrt{3}$
   Product, $3\sqrt{3} + \sqrt{15}$.

2. Multiply $3 + \sqrt{5}$
   by $2 + \sqrt{3}$
   Product, $6 + 2\sqrt{5} + 3\sqrt{3} + \sqrt{15}$.

3. Multiply $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$
   by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$

$$\begin{array}{l} a + a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} \\ \quad - a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} - b \\ \hline \end{array}$$

   Product, $a - b$.

Multiply

4. $\sqrt{5} + \sqrt{3}$ by $\sqrt{5} - \sqrt{3}$. *Ans.* 2.

5. $a + 3\sqrt{b}$ by $a - 2\sqrt{b}$. *Ans.* $a^2 + a\sqrt{b} - 6b$.

6. $x + 2\sqrt{y} + 3\sqrt[3]{z}$ by $x - 2\sqrt{y} + 3\sqrt[3]{z}$.
   Ans. $x^2 - 4y + 6x\sqrt[3]{z} + 9\sqrt[3]{z^2}$.

7. $a^{\frac{3}{2}} + 2b^{\frac{2}{3}}$ by $a^{\frac{1}{2}} - 4b^{\frac{1}{2}}$.
   Ans. $a^2 + 2a^{\frac{1}{2}}b^{\frac{2}{3}} - 4a^{\frac{3}{2}}b^{\frac{1}{2}} - 8b^{\frac{7}{6}}$.

8. $\sqrt{a} + \sqrt{b + x}$ by $\sqrt{a} - \sqrt{b + x}$.
   Ans. $a - b - x$.

9. $\sqrt{a} - \sqrt[3]{a}$ by $\sqrt[3]{a} - 1$.
   Ans. $\sqrt[6]{a^5} - \sqrt[3]{a^2} - \sqrt{a} + \sqrt[3]{a}$.

10. $a^{\frac{5}{2}} + a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} + ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{3}}$.
    Ans. $a^3 - b^2$.

11. $x^{\frac{3}{2}} - 4ax^{\frac{1}{2}} + 2a^{\frac{3}{2}}$ by $x - a^{\frac{1}{2}}x^{\frac{1}{2}}$.
    Ans. $x^{\frac{5}{2}} - a^{\frac{1}{2}}x^2 - 4ax^{\frac{3}{2}} + 6a^{\frac{3}{2}}x - 2a^2x^{\frac{1}{2}}$.

12. $a^{\frac{3}{2}} - a^{\frac{2}{3}} + a^{-\frac{1}{3}} - a^{-\frac{3}{2}}$ by $a^{\frac{3}{2}} + a^{\frac{2}{3}} - a^{-\frac{1}{3}} - a^{-\frac{3}{2}}$.

DIVISION.

246. To find the quotient of two simple radicals.

The rules for finding the quotient of two simple radicals may be derived from those for finding their product, by substituting the word *quotient* for the word *product*, and Art. **235** for Art. **234**. Let the student repeat those rules with this modification.

Cor. If the radicals are expressed by means of fractional exponents, their quotient may be found by the rule of Art. **85**. Thus, $\frac{6a^{\frac{5}{6}}}{3a^{\frac{1}{3}}} = 2a^{\frac{5}{6}-\frac{1}{3}} = 2a^{\frac{1}{2}}$.

EXERCISES.

Divide

1. $45\sqrt{6}$ by $5\sqrt{9}$. *Ans.* $3\sqrt{6}$.
2. 60 by $6\sqrt{5}$. *Ans.* $2\sqrt{5}$.
3. $4\sqrt[3]{15}$ by $\frac{1}{2}\sqrt[3]{180}$. *Ans.* $\frac{4}{3}\sqrt[3]{18}$.
4. $10\sqrt[6]{432}$ by $5\sqrt{3}$. *Ans.* $2\sqrt[3]{4}$.
5. $12ab\sqrt{2}$ by $\sqrt{12b}$. *Ans.* $2a\sqrt{6b}$.
6. $ab\sqrt[6]{a^2b^3x^2y^3}$ by $b\sqrt{by}$. *Ans.* $a\sqrt[3]{ax}$.
7. $15\sqrt[12]{a^3b^4}$ by $3\sqrt[4]{a}$. *Ans.* $5\sqrt[3]{b}$.
8. $a^{\frac{5}{2}}$ by $a^{\frac{3}{4}}$. *Ans.* $a^{\frac{7}{4}}$.
9. $6a^{\frac{3}{4}}bcd^{\frac{3}{5}}$ by $3a^{\frac{1}{4}}b^{\frac{1}{3}}c^{\frac{1}{5}}d^{\frac{3}{5}}$. *Ans.* $2a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{4}{5}}$.
10. $mn(a^2-b^2)^{\frac{2}{3}}$ by $n(a-b)^{\frac{2}{3}}$. *Ans.* $m(a+b)^{\frac{2}{3}}$.
11. $2(x+y)^{-\frac{2}{3}}$ by $(x+y)^{\frac{1}{3}}$. *Ans.* $\frac{2}{x+y}$.
12. $6ab^{\frac{2}{3}}c^2d^{\frac{3}{5}}$ by $-3a^{\frac{1}{2}}b^{-\frac{1}{3}}cd^{-\frac{2}{5}}$. *Ans.* $-2a^{\frac{1}{2}}bcd$.

247. To find the quotient of two quantities, when one or each of them is a polynomial radical.

The quotient of two quantities, when one or each of them is a polynomial radical, may be found by combining the rules of Art. **246** with those of Articles **87** and **88**. If the radicals are expressed by means of fractional exponents, the rules of Articles **87** and **88** are sufficient.

EXERCISES.

1. Divide $a^2 + a\sqrt{b} - 6b$ by $a - 2\sqrt{b}$.

$$\begin{array}{l|l}
a^2 + a\sqrt{b} - 6b & a - 2\sqrt{b} \\ \cline{2-2}
\underline{a^2 - 2a\sqrt{b}} & a + 3\sqrt{b}. \\
\phantom{a^2 + {}}3a\sqrt{b} - 6b & \\
\phantom{a^2 + {}}\underline{3a\sqrt{b} - 6b} &
\end{array}$$

2. Divide $x^{\frac{1}{2}} - y^{\frac{1}{2}}$ by $x^{\frac{1}{4}} - y^{\frac{1}{4}}$.

$$\begin{array}{l|l}
x^{\frac{1}{2}} - y^{\frac{1}{2}} & x^{\frac{1}{4}} - y^{\frac{1}{4}} \\ \cline{2-2}
\underline{x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}}} & x^{\frac{1}{4}} + y^{\frac{1}{4}}. \\
x^{\frac{1}{4}}y^{\frac{1}{4}} - y^{\frac{1}{2}} & \\
\underline{x^{\frac{1}{4}}y^{\frac{1}{4}} - y^{\frac{1}{2}}} &
\end{array}$$

Divide

3. $8\sqrt{108} + 3\sqrt{18} - 5\sqrt{24}$ by $\sqrt{6}$.
Ans. $24\sqrt{2} + 3\sqrt{3} - 10$.

4. $4\sqrt{50} - 2\sqrt{10} + 4\sqrt{15}$ by $2\sqrt{5}$.
Ans. $2\sqrt{10} - \sqrt{2} + 2\sqrt{3}$.

5. $\sqrt{x^5} - 4\sqrt{x^3} - 2\sqrt{x} + 6x - x^2$ by $\sqrt{x^3} + 2 - 4\sqrt{x}$.
Ans. $x - \sqrt{x}$.

6. $a^2 - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{6}}$.
Ans. $a^{\frac{5}{3}} + a^{\frac{4}{3}}b^{\frac{1}{6}} + ab^{\frac{1}{3}} + a^{\frac{2}{3}}b^{\frac{1}{2}} + a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{5}{6}}$.

7. $a - b^2$ by $a^{\frac{3}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b + b^{\frac{3}{2}}$. *Ans.* $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

8. $x^{-1} - y^{-1}$ by $x^{-\frac{1}{3}} - y^{-\frac{1}{3}}$.
Ans. $x^{-\frac{2}{3}} + x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$.

9. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$. *Ans.* $x + y$.

10. $x^{\frac{3}{2}} + y^{\frac{3}{2}}$ by $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y$. *Ans.* $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.

11. $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{1}{2}} + y$ by $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.
Ans. $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

12. $x^{\frac{3}{4}} - a^3$ by $x^{\frac{1}{4}} - a^{\frac{1}{2}}x^{\frac{1}{8}} + a$.
Ans. $x^{\frac{1}{2}} + a^{\frac{1}{2}}x^{\frac{3}{8}} - a^{\frac{3}{2}}x^{\frac{1}{8}} - a^2$.

13. $a^{\frac{5}{2}} - a^{\frac{3}{2}}b + ab^{\frac{3}{2}} - 2a^{\frac{1}{2}}b^2 + b^{\frac{5}{2}}$ by $a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}$.
Ans. $a + a^{\frac{1}{2}}b^{\frac{1}{2}} - b$.

14. $x^3 - 4x^2 + 4x^{\frac{3}{2}} - x$ by $x^{\frac{3}{2}} + 2x - x^{\frac{1}{2}}$.
Ans. $x^{\frac{3}{2}} - 2x + x^{\frac{1}{2}}$.

INVOLUTION.

248. TH. *The m^{th} power of the n^{th} root of a quantity is equal to the n^{th} root of the m^{th} power of that quantity.*

Thus,
$$(\sqrt[3]{3})^2 = \sqrt[3]{3^2};$$
for
$$(\sqrt[3]{3})^2 = \sqrt[3]{3} \times \sqrt[3]{3} = \sqrt[3]{3^2} \text{ (234)}.$$

In like manner, it may be shown that
$$(\sqrt[3]{3})^3 = \sqrt[3]{3^3} = 3;$$
$$(\sqrt[3]{3})^4 = \sqrt[3]{3^4};$$
$$(\sqrt[3]{3})^5 = \sqrt[3]{3^5}; \quad \text{and so on.}$$

COR. 1. If the index of the root is equal to the exponent of the power to which the root is to be raised, the required power may be found by simply removing the radical sign. Thus, $(\sqrt{3})^2 = 3$, $(\sqrt[3]{3})^3 = 3$, $(\sqrt[4]{3})^4 = 3$.

Cor. 2. If the root is indicated by a fractional exponent, the rule of Art. **202** is applicable. Thus,

$$(a^{\frac{1}{2}})^3 = a^{\frac{3}{2}}, \quad (a^{\frac{1}{3}})^2 = a^{\frac{2}{3}}, \quad (a^{\frac{1}{4}})^3 = a^{\frac{3}{4}}.$$

249. To find any power of a radical.

Any power of a radical may be found by combining the principle of Art. **248** with the rules for involution in Chapter IX. If each root in the given radical is indicated by a fractional exponent, the rules for involution in Chapter IX are sufficient.

EXERCISES.

1. Find the square of $3\sqrt[3]{5}$.

$$(3\sqrt[3]{5})^2 = 3^2(\sqrt[3]{5})^2 = 3^2\sqrt[3]{5^2} = 9\sqrt[3]{25} \ \textbf{(248)}.$$

2. Find the square of $\sqrt{3} + \sqrt{5}$.

$$(\sqrt{3} + \sqrt{5})^2 = (\sqrt{3})^2 + 2\sqrt{3} \times \sqrt{5} + (\sqrt{5})^2$$
$$= 3 + 2\sqrt{15} + 5 = 8 + 2\sqrt{15} \ \textbf{(81)}.$$

3. Find the square of $5\sqrt[3]{a}$. *Ans.* $25\sqrt[3]{a^2}$.

4. Find the cube of $-5a\sqrt[3]{x}$. *Ans.* $-125a^3x$.

5. Find the cube of $\frac{2}{3}\sqrt{3}$. *Ans.* $\frac{8}{9}\sqrt{3}$.

6. Find the 4th power of $-a^{\frac{2}{3}}$. *Ans.* $a^{\frac{8}{3}}$.

7. Find the square of $a\sqrt{2} - \sqrt{3}$.
Ans. $2a^2 - 2a\sqrt{6} + 3$.

8. Find the cube of $3 - \sqrt{5}$. *Ans.* $72 - 32\sqrt{5}$.

9. Find the square of $a^{\frac{1}{2}} - b^{\frac{1}{3}}$. *Ans.* $a - 2a^{\frac{1}{2}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

10. Find the 4th power of $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

Ans. $a^2 + 4a^{\frac{3}{2}}b^{\frac{1}{2}} + 6ab + 4a^{\frac{1}{2}}b^{\frac{3}{2}} + b^2$.

11. Find the cube of $a^{\frac{2}{3}} - 2a^{\frac{1}{3}}b^{\frac{1}{6}} + 3b^{\frac{1}{3}}$.

Ans. $a^2 - 6a^{\frac{5}{3}}b^{\frac{1}{6}} + 21a^{\frac{4}{3}}b^{\frac{1}{3}} - 44ab^{\frac{1}{2}} + 63a^{\frac{2}{3}}b^{\frac{2}{3}} - 54a^{\frac{1}{3}}b^{\frac{5}{6}} + 27b$.

12. Find the 5th power of $x^{\frac{3}{4}} - y^{\frac{5}{2}}$.

Ans. $x^{\frac{15}{4}} - 5x^3y^{\frac{5}{2}} + 10x^{\frac{9}{4}}y^5 - 10x^{\frac{3}{2}}y^{\frac{15}{2}} + 5x^{\frac{3}{4}}y^{10} - y^{\frac{25}{2}}$.

EVOLUTION.

250. To find any root of a radical.

Any root of a radical may be found by combining the principle of Art. **217** with the rules for evolution in Chapter IX.

If each root in the given radical is indicated by a fractional exponent, the rules for evolution in Chapter IX are sufficient.

EXERCISES.

1. Find the square root of $9\sqrt[3]{3^2}$.

$$\sqrt{9\sqrt[3]{3^2}} = 3\sqrt{\sqrt[3]{3^2}} = 3\sqrt[6]{3^2}$$

$$= 3\sqrt[3]{3} \text{ (\textbf{217}, \textbf{241}, Cor. 2).}$$

2. Find the square root of $a + 2\sqrt{ab} + b$.

$$\begin{array}{r|l|l} & a + 2\sqrt{ab} + b & \underline{\sqrt{a} + \sqrt{b}.} \\ & a & \\ \hline 2\sqrt{a} + \sqrt{b} & 2\sqrt{ab} + b & \\ & \underline{2\sqrt{ab} + b} & \end{array}$$

3. Find the square root of $25\sqrt[3]{3}$. *Ans.* $5\sqrt[6]{3}$.

4. Find the square root of $2\sqrt[3]{5}$. *Ans.* $\sqrt[6]{40}$.

5. Find the cube root of $a\sqrt{a}$. *Ans.* $\sqrt{a}$.

6. Find the fourth root of $5\sqrt[3]{9}$. *Ans.* $\sqrt[12]{1125}$.

7. Find the square root of $4\sqrt[3]{a^2} + 12\sqrt[3]{ab} + 9\sqrt[3]{b^2}$.
Ans. $2\sqrt[3]{a} + 3\sqrt[3]{b}$.

8. Find the square root of $x^{\frac{1}{2}} + 2x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.
Ans. $x^{\frac{1}{4}} + y^{\frac{1}{4}}$.

9. Find the square root of $a^{\frac{1}{3}} - 2a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}$.
Ans. $a^{\frac{1}{6}} - b^{\frac{1}{6}}$.

10. Find the cube root of $a + 3a^{\frac{2}{3}}b^{\frac{1}{3}} + 3a^{\frac{1}{3}}b^{\frac{2}{3}} + b$.
Ans. $a^{\frac{1}{3}} + b^{\frac{1}{3}}$

11. Find the cube root of $a^{-\frac{3}{2}}x^{\frac{3}{2}} - 3a^{-1}x + 6a^{-\frac{1}{2}}x^{\frac{1}{2}} - 7 + 6a^{\frac{1}{2}}x^{-\frac{1}{2}} - 3ax^{-1} + a^{\frac{3}{2}}x^{-\frac{3}{2}}$. *Ans.* $a^{-\frac{1}{2}}x^{\frac{1}{2}} - 1 + a^{\frac{1}{2}}x^{-\frac{1}{2}}$.

12. Find the fourth root of $16x^6 - 96x^{\frac{9}{2}}y^{\frac{3}{4}} + 216x^3y^{\frac{3}{2}} - 216x^{\frac{3}{2}}y^{\frac{9}{4}} + 81y^3$. *Ans.* $2x^{\frac{3}{2}} - 3y^{\frac{3}{4}}$.

REDUCTION OF FRACTIONS

HAVING SURD DENOMINATORS TO OTHERS HAVING RATIONAL DENOMINATORS.

251. To reduce a fraction whose denominator is a surd of the second degree to another having a rational denominator.

1. Let it be required to reduce the fraction $\frac{3}{\sqrt{2}}$ to another having a rational denominator.

$$\frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

2. Let it be required to reduce the fraction $\frac{6}{\sqrt{7}+\sqrt{2}}$ to another having a rational denominator.

$$\frac{6}{\sqrt{7}+\sqrt{2}}=\frac{6(\sqrt{7}-\sqrt{2})}{(\sqrt{7}+\sqrt{2})(\sqrt{7}-\sqrt{2})}=\frac{6(\sqrt{7}-\sqrt{2})}{7-2}$$
$$=\frac{6(\sqrt{7}-\sqrt{2})}{5}.$$

RULES.

I. *If the denominator is a simple surd of the second degree, multiply both terms of the fraction by the denominator.*

II. *If the denominator is a binomial surd of the second degree, multiply both terms of the fraction by a binomial which differs from the given denominator only in the sign of the second term.*

EXERCISES.

Reduce each of the following fractions to another having a rational denominator:

1. $\frac{2}{\sqrt{5}}$. *Ans.* $\frac{2\sqrt{5}}{5}$.
2. $\frac{\sqrt{3}}{2\sqrt{6}}$. *Ans.* $\frac{\sqrt{2}}{4}$.
3. $\frac{3a\sqrt{2}}{2\sqrt{3}}$. *Ans.* $\frac{a\sqrt{6}}{2}$.
4. $\frac{a}{\sqrt{b}}$. *Ans.* $\frac{a\sqrt{b}}{b}$.
5. $\frac{3}{\sqrt{5}-\sqrt{2}}$. *Ans.* $\sqrt{5}+\sqrt{2}$.
6. $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$. *Ans.* $\frac{7-2\sqrt{10}}{3}$.

7. $\dfrac{a + \sqrt{b}}{a - \sqrt{b}}$. *Ans.* $\dfrac{a^2 + b + 2a\sqrt{b}}{a^2 - b}$.

8. $\dfrac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$. *Ans.* $\dfrac{a + b - 2\sqrt{ab}}{a - b}$.

9. $\dfrac{a}{3 + \sqrt{a}}$. *Ans.* $\dfrac{3a - a\sqrt{a}}{9 - a}$.

10. $\dfrac{a}{a + \sqrt{1 + a^2}}$. *Ans.* $a\sqrt{1 + a^2} - a^2$.

252. To reduce a fraction whose denominator is a simple surd of any degree to another having a rational denominator.

Let it be required to reduce $\dfrac{\sqrt{5}}{3\sqrt[3]{9}}$ to a fraction having a rational denominator.

$$\frac{\sqrt{5}}{3\sqrt[3]{9}} = \frac{\sqrt{5} \times \sqrt[3]{3}}{3\sqrt[3]{9} \times \sqrt[3]{3}} = \frac{\sqrt[6]{125} \times \sqrt[6]{9}}{3\sqrt[3]{27}} = \frac{\sqrt[6]{1125}}{9}.$$

RULE.

Multiply both terms of the fraction by such a quantity as will make the quantity under the radical sign in the denominator a perfect power corresponding to the degree of the indicated root in that term.

SCH. The multiplier should be as small as possible.

EXERCISES.

Reduce each of the following fractions to another having a rational denominator:

1. $\dfrac{2}{\sqrt[3]{3}}$. *Ans.* $\dfrac{2\sqrt[3]{9}}{3}$.

2. $\dfrac{2}{3\sqrt[3]{a^2}}$. *Ans.* $\dfrac{2\sqrt[3]{a}}{3a}$.

3. $\dfrac{3}{\sqrt[3]{4}}$. *Ans.* $\dfrac{3\sqrt[3]{2}}{2}$.

4. $\dfrac{\sqrt[4]{2}}{\sqrt[4]{3}}$. *Ans.* $\dfrac{\sqrt[4]{54}}{3}$.

5. $\dfrac{a}{b\sqrt[3]{c}}$. *Ans.* $\dfrac{a\sqrt[3]{c^2}}{bc}$.

6. $\dfrac{a}{b\sqrt[4]{c}}$. *Ans.* $\dfrac{a\sqrt[4]{c^3}}{bc}$.

7. $\dfrac{a}{b\sqrt[4]{c^3}}$. *Ans.* $\dfrac{a\sqrt[4]{c}}{bc}$.

8. $\dfrac{a}{b\sqrt[5]{c^4}}$. *Ans.* $\dfrac{a\sqrt[5]{c}}{bc}$.

9. $\dfrac{a}{bc^{\frac{1}{m}}}$. *Ans.* $\dfrac{ac^{\frac{m-1}{m}}}{bc}$.

10. $\dfrac{a^{\frac{1}{m}}}{ab^{\frac{1}{n}}}$. *Ans.* $\dfrac{a^{\frac{1}{m}}b^{\frac{n-1}{n}}}{ab}$.

11. $\dfrac{a+x}{(a-x)^{\frac{1}{2}}}$. *Ans.* $\dfrac{(a+x)(a-x)^{\frac{1}{2}}}{a-x}$.

12. $\dfrac{a-x}{(a+x)^{\frac{2}{3}}}$. *Ans.* $\dfrac{(a-x)(a+x)^{\frac{1}{3}}}{a+x}$.

IMAGINARY QUANTITIES.

253. Imaginary quantities are to be treated as other radicals; but, in multiplication and division, a slight modification of their forms is sometimes necessary to determine the sign of the result. This will be made plain in the following

EXERCISES.

1. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$.

$$2\sqrt{-3} = 2\sqrt{3(-1)} = 2\sqrt{3} \times \sqrt{-1};$$
$$3\sqrt{-2} = 3\sqrt{2(-1)} = 3\sqrt{2} \times \sqrt{-1};$$

$$\therefore\ (2\sqrt{-3})(3\sqrt{-2}) = 2\sqrt{3} \times \sqrt{-1} \times 3\sqrt{2} \times \sqrt{-1}$$
$$= 6\sqrt{6} \times (-1) = -6\sqrt{6}.$$

2. Multiply $b\sqrt{-a}$ by $c\sqrt{-a}$.

It is evident that $(b\sqrt{-a})(c\sqrt{-a}) = bc(\sqrt{-a})^2$ $= bc(-a) = -abc$.

It is also evident that $(b\sqrt{-a})(c\sqrt{-a}) = bc\sqrt{a^2}$; hence, $\sqrt{a^2} = -a$.

The ambiguity with regard to the sign of $\sqrt{a^2}$ is removed by remembering that a^2 was obtained by squaring $-a$. If it were not known in what way a^2 was produced, then the sign of its square root would be ambiguous.

3. Multiply $\sqrt{-a}$ by $\sqrt{-b}$.

$$\sqrt{-a} = \sqrt{a \times (-1)} = \sqrt{a} \times \sqrt{-1},$$
$$\sqrt{-b} = \sqrt{b \times (-1)} = \sqrt{b} \times \sqrt{-1};$$
$$\therefore\ \sqrt{-a} \times \sqrt{-b} = \sqrt{a} \times \sqrt{-1} \times \sqrt{b} \times \sqrt{-1}$$
$$= \sqrt{ab} \times (\sqrt{-1})^2 = \sqrt{ab} \times (-1) = -\sqrt{ab}.$$

Hence, the ambiguity with regard to the sign of the product of two simple imaginary quantities may be removed by reducing each factor to the form of $a\sqrt{-1}$ and observing that $(\sqrt{-1})^2 = -1$.

4. Divide $\sqrt{a}$ by $\sqrt{-b}$.

$$\frac{\sqrt{a}}{\sqrt{-b}} = \frac{\sqrt{a}}{\sqrt{b} \times \sqrt{-1}} = \frac{1}{\sqrt{-1}} \times \frac{\sqrt{a}}{\sqrt{b}}. \qquad (1)$$

But $\sqrt{-1} \times \sqrt{-1} = -1$;

whence, $-\sqrt{-1} \times \sqrt{-1} = 1$. (2)

Dividing both members of (2) by $\sqrt{-1}$,

$$-\sqrt{-1} = \frac{1}{\sqrt{-1}}. \qquad (3)$$

Substituting $-\sqrt{-1}$ for $\dfrac{1}{\sqrt{-1}}$ in (1),

$$\frac{\sqrt{a}}{\sqrt{-b}} = -\sqrt{-1} \times \frac{\sqrt{a}}{\sqrt{b}} = -\sqrt{-1} \times \sqrt{\frac{a}{b}}$$

$$= -\sqrt{-\frac{a}{b}}.$$

5. Multiply $\quad 5 + 2\sqrt{-1}$

by $\quad 5 - 2\sqrt{-1}$

$$\begin{array}{l}
25 + 10\sqrt{-1} \\
\quad\;\; - 10\sqrt{-1} - 4 \times (-1) \\
\hline
25 \qquad\qquad\qquad + 4 = 29
\end{array}$$

The same result may be obtained by the principle of Art. **83**. Thus $5^2 = 25$, and $(2\sqrt{-1})^2 = -4$; hence the product is $25 - (-4)$, or 29.

6. Find the sum of $\sqrt{-1}$, $\sqrt{-9}$, and $\sqrt{-16}$.
Ans. $8\sqrt{-1}$.

7. Find the sum of $\sqrt{-1}$, $\sqrt{-2}$, and $\sqrt{-3}$.
Ans. $(1 + \sqrt{2} + \sqrt{3})\sqrt{-1}$.

8. Subtract $2\sqrt{-4}$ from $\sqrt{-36}$. *Ans.* $2\sqrt{-1}$.

9. Subtract $c\sqrt{-d}$ from $a\sqrt{-d}$.
Ans. $(a - c)\sqrt{-d}$.

10. Subtract $c\sqrt{-d}$ from $a\sqrt{-b}$.
Ans. $(a\sqrt{b} - c\sqrt{d})\sqrt{-1}$.

11. Multiply $2\sqrt{-4}$ by $3\sqrt{-2}$. *Ans.* $-12\sqrt{2}$.

12. Multiply $a\sqrt{-bc}$ by $y\sqrt{-x}$. *Ans.* $-ay\sqrt{bcx}$.

13. Multiply $2 - \sqrt{-2}$ by $1 + \sqrt{-2}$.
Ans. $4 + \sqrt{-2}$.

14. Multiply $6 - \sqrt{-4}$ by $6 + \sqrt{-4}$. *Ans.* 40.

15. Multiply $a + b\sqrt{-1}$ by $a - b\sqrt{-1}$.
Ans. $a^2 + b^2$.

16. Reduce $\dfrac{2}{5 + \sqrt{-3}}$ to a fraction whose denominator shall be rational. *Ans.* $\dfrac{5 - \sqrt{-3}}{14}$.

17. Divide $6\sqrt{6}$ by $3\sqrt{-2}$. *Ans.* $-2\sqrt{-3}$.

18. Divide $6\sqrt{-3}$ by $\sqrt{-1}$. *Ans.* $6\sqrt{3}$.

19. Divide $4 + \sqrt{-2}$ by $1 + \sqrt{-2}$.
Ans. $2 - \sqrt{-2}$.

20. Divide $x^2 + a^2$ by $x - a\sqrt{-1}$.
Ans. $x + a\sqrt{-1}$.

RADICAL EQUATIONS.

254. A **Radical Equation** is an equation in which an unknown quantity occurs under a radical sign. Thus, $x + \sqrt{x} = 12$, and $x + \sqrt{y} - \sqrt{xy} = 25$ are radical equations.

EXERCISES.

1. Solve $\sqrt{x} - \sqrt{3} = 2$.

By transposition, $\sqrt{x} = 2 + \sqrt{3}$;

whence,
$$x = (2 + \sqrt{3})^2 = 4 + 4\sqrt{3} + 3$$
$$= 7 + 4\sqrt{3}.$$

2. Solve $\dfrac{3x - 1}{\sqrt{3x} + 1} = \dfrac{\sqrt{3x} - 1}{2} + 1$.

Performing the division indicated in the first member, the equations become

$$\sqrt{3x} - 1 = \frac{\sqrt{3x} - 1}{2} + 1;$$

whence, $$2\sqrt{3x} - 2 = \sqrt{3x} - 1 + 2;$$

whence, $$\sqrt{3x} = 3; \quad \text{whence, } x = 3.$$

Solve:

3. $\sqrt{21 + x} = 3 + \sqrt{x}$. *Ans.* $x = 4$.

4. $2 + \sqrt{x} = \sqrt{12 + x}$. *Ans.* $x = 4$.

5. $\sqrt{5x + 10} - \sqrt{5x} = 2$. *Ans.* $x = \frac{9}{20}$.

6. $\sqrt{9x} + 5 = \sqrt{108 + 9x}$. *Ans.* $x = \left(\frac{83}{30}\right)^2$.

7. $\sqrt[3]{17x - 6} = 2$. *Ans.* $x = \frac{14}{17}$.

8. $\sqrt{x} - \sqrt{x - 2} = \sqrt{2}$. *Ans.* $x = 2$.

9. $\sqrt{x - 5} + \sqrt{x - 12} = 7$. *Ans.* $x = 21$.

10. $\sqrt{x + 1} + \sqrt{x} = \frac{1}{\sqrt{x}}$. *Ans.* $x = \frac{1}{3}$.

11. $\frac{x(1 - a)}{\sqrt{x}} = \frac{\sqrt{x}}{x}$. *Ans.* $x = \frac{1}{1 - a}$.

12. $(a - x)^{\frac{1}{2}} = (a + x^2)^{\frac{1}{4}}$. *Ans.* $x = \frac{a - 1}{2}$.

13. $\sqrt{\frac{a^2}{x}} + \sqrt{\frac{x}{c^2}} = \sqrt{\frac{n}{x}}$. *Ans.* $x = c\left(\sqrt{n} - a\right)$.

14. $\sqrt{x} - \sqrt{\frac{a}{x}} = \sqrt{a + x}$. *Ans.* $x = \frac{a}{a + 2\sqrt{a}}$.

CHAPTER XI.

EQUATIONS.

255. The methods of solving simple equations have already been given (Chapter VII). The object of the present chapter is to explain *the methods of solving* (1) *a quadratic equation containing only one unknown quantity;* (2) *a higher equation containing only one unknown quantity;* (3) *a group of simultaneous equations, at least one of which is of a higher degree than the first.*

QUADRATIC EQUATIONS

CONTAINING ONLY ONE UNKNOWN QUANTITY.

256. A **Quadratic Equation** is an equation which, in its simplest form, contains the square of the unknown quantity and no higher power. Thus, $3x^2 = 27$ and $3x^2 + 5x = 2$ are quadratic equations.

257. A quadratic equation having only one unknown quantity, as x, cannot have more than three kinds of terms: (1) terms which contain x^2; (2) terms which contain x; (3) terms independent of x, that is, known terms.

258. A quadratic equation which has the three kinds of terms just described is said to be *complete;* otherwise, it is said to be *incomplete.* Thus, $7x^2 = 3x + 160$ is complete; but $3x^2 = 27$ and $3x^2 = 9x$ are incomplete.

A complete quadratic equation is sometimes called an

Affected Quadratic, and an incomplete equation of the form of $3x^2 = 27$ is sometimes called a *Pure Quadratic.*

259. The general principles and processes explained in the chapter for the solution of simple equations are applicable to quadratics.

Axiom 5, in its relation to quadratics, may be stated thus:

The square roots (of like signs) of equals are equal.

PURE QUADRATICS.

260. To solve a pure quadratic equation.

Let it be required to solve the equation

$$x^2 + 5 = \frac{10x^2}{3} - 16.$$

By transposition and reduction,

$$x^2 = \frac{10x^2}{3} - 21.$$

Multiplying by 3, $3x^2 = 10x^2 - 63$;

whence, $7x^2 = 63$;

whence, $x^2 = 9$.

Extracting the square root of both members (**64**, 5),

$$x = \pm 3.$$

Sch. 1. The complete solution of an equation consists in finding all possible values of x which will satisfy the equation. This equation is satisfied by $+3$ and by -3, for $(+3)^2 = 9$, and $(-3)^2 = 9$. Hence, both values must be given, and this is done by placing the double sign ± before the 3.

No greater generality would be gained by writing $\pm x = \pm 3$; therefore, the double sign is confined to the second member.

SCH. 2. Axiom 5 does not imply that every square root of a number is equal to every square root of its equal. Thus, if the square roots of both members of the equation $9 = 9$ be extracted, this axiom contains no authority for writing $+3 = -3$. It was intended to exclude such results as this by saying that *like* roots of equals are equal.

SCH. 3. Every quadratic equation has two roots; that is, the unknown quantity has two values which, separately, satisfy the equation.

EXERCISES.

Solve:

1. $3x^2 + 12 = 687$. *Ans.* $x = \pm 15$.
2. $5x^2 - 125 = 0$. *Ans.* $x = \pm 5$.
3. $(x - 5)(x - 3) = 16 - 8x$. *Ans.* $x = \pm 1$.
4. $(3x - 4)(2x - 7) + 29x = 244$. *Ans.* $x = \pm 6$.
5. $30(x^2 - 3) = 6x^2 + 6$. *Ans.* $x = \pm 2$.
6. $x^2 - \frac{3x^2}{4} - 9 = 0$. *Ans.* $x = \pm 6$.
7. $\frac{3x^2 - 4}{2x^2 - 3} = \frac{6x^2 - 7}{4x^2 - 5}$. *Ans.* $x = \pm 1$.
8. $\frac{x^2 + x}{2} - \frac{x^2 - x + 16}{3} = \frac{5x - x^2}{6}$. *Ans.* $x = \pm 4$.
9. $\frac{5}{2x^2 + 1} = \frac{2}{5x^2 - 8}$. *Ans.* $x = \pm \sqrt{2}$.
10. $\frac{x^2 + 7}{11} - \frac{2x^2 - 16}{3} + \frac{2x^2 + 5}{4} = 5\frac{1}{3} + \frac{3x^2 + 7}{12}$.
    Ans. $x = \pm 2$.
11. $\frac{3x^2 - 3}{4} - \frac{3x^2 - 4}{3} = 5\frac{1}{3} - \frac{27 + 4x^2}{9}$.
    Ans. $x = \pm 3$.
12. $\frac{a}{x} + \frac{\sqrt{a^2 - x^2}}{x} - \frac{x}{b} = 0$.
    Ans. $x = \pm \sqrt{2ab - b^2}$.

13. $ax^2 - b = c.$ *Ans.* $x = \pm\sqrt{\dfrac{b+c}{a}}.$

14. $2ax^2 + b - 4 = cx^2 - 5 + d - ax^2.$

Ans. $x = \pm\sqrt{\dfrac{d-b-1}{3a-c}}.$

15. $\dfrac{a}{x} + \dfrac{a^2 - x^2}{x} = \dfrac{x}{b}.$ *Ans.* $x = \pm\sqrt{\dfrac{ab(1+a)}{1+b}}.$

16. $\sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2+x^2}}.$ *Ans.* $x = \pm a.$

17. $\dfrac{2x + \sqrt{4x^2-1}}{2x - \sqrt{4x^2-1}} = 4.$ *Ans.* $x = \pm\dfrac{5}{8}.$

18. $\dfrac{x}{\sqrt{1-x}+1} + \dfrac{x}{\sqrt{1+x}-1} = 1.$ *Ans.* $x = \pm\dfrac{\sqrt{3}}{2}.$

COMPLETE QUADRATICS.

261. To solve a complete quadratic equation by completing the square.

1. Let it be required to solve the equation

$$x^2 + 2ax = 3a^2.$$

It is evident that the first member of this equation will become a perfect square by the addition of a^2 as a third term.

Adding, then, a^2 to both members, it becomes

$$x^2 + 2ax + a^2 = 4a^2,$$

or
$$(x + a)^2 = 4a^2.$$

Extracting the square root of both members,

$$x + a = \pm 2a;$$

whence, $x = \pm 2a - a = a$ or $-3a.$

The quantity a^2, added to the expression $x^2 + 2ax$ in order to make the first member a *perfect square*, is the square of half the coefficient of x. The operation by which the first member is made a perfect square is called *completing the square.*

2. Let it be required to solve the equation

$$x^2 + ax = b.$$

Completing the square,

$$x^2 + ax + \frac{a^2}{4} = b + \frac{a^2}{4}.$$

Extracting the square root of both members,

$$x + \frac{a}{2} = \pm\sqrt{b + \frac{a^2}{4}};$$

whence, $$x = -\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}.$$

The equation $x^2 + ax = b$, in which a and b may be either positive or negative, integral or fractional, may be considered as *the standard form* for complete quadratics, for all such equations may be reduced to that form. It should be observed that, in the standard form, the coefficient of x^2 is $+1$.

RULE.

1. *Reduce the given equation to the standard form.*
2. *COMPLETE THE SQUARE by adding to both members the SQUARE OF HALF the coefficient of x.*
3. *Extract the square root of both members, prefix the sign $\pm$ to the second member, and solve the result as a simple equation.*

EXERCISES.

1. Solve $$3x^2 + 18x + 4 = 52.$$

By transposition and reduction,

$$3x^2 + 18x = 48.$$

By division, $$x^2 + 6x = 16.$$

The coefficient of x is 6, the half of which is 3.

Adding the square of 3 to both members, to *complete the square,*

$$x^2 + 6x + 3^2 = 16 + 9 = 25.$$

Extracting the square root of both members,

$$x + 3 = \pm 5;$$

whence, $x = \pm 5 - 3 = 2$ or -8.

2. Solve: $-6x^2 + 72x = 210.$

Dividing both members by -6,

$$x^2 - 12x = -35.$$

Completing the square,

$$x^2 - 12x + 36 = -35 + 36 = 1.$$

Extracting the square root of both members,

$$x - 6 = \pm 1;$$

whence, $x = \pm 1 + 6 = 7$ or 5.

Solve:

3. $x^2 - 8x + 15 = 0.$ *Ans.* $x = 5$ or 3.

4. $x^2 + 12x = -35.$ *Ans.* $x = -5$ or -7.

5. $x^2 - x = 6.$ *Ans.* $x = 3$ or -2.

6. $x^2 + 2x = 35.$ *Ans.* $x = 5$ or -7.

7. $x^2 + 2x - 24 = 0.$ *Ans.* $x = 4$ or -6.

8. $x^2 + 9x = 22.$ *Ans.* $x = 2$ or -11.

9. $6x^2 + x = 35.$ *Ans.* $x = 2\frac{1}{3}$ or $-2\frac{1}{2}$.

10. $6x^2 - 15x = 56 - 10x.$ *Ans.* $x = 3\frac{1}{2}$ or $-2\frac{2}{3}$.

11. $x^2 + 12x = 85.$ *Ans.* $x = 5$ or -17.

12. $x^2 - 2x = 24.$ *Ans.* $x = 6$ or -4.

13. $x^2 - \frac{x}{3} = 3\frac{1}{3}.$ *Ans.* $x = 2$ or $-1\frac{2}{3}$.

14. $5x - 2x^2 = 2.$ *Ans.* $x = 2$ or $\frac{1}{2}$.

15. $a + bx + cx^2 = 0$. *Ans.* $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$.

16. $acx^2 + adx = bcx + bd$. *Ans.* $x = \frac{b}{a}$ or $-\frac{d}{c}$.

17. $x^2 + (a - b)x = ab$. *Ans.* $x = b$ or $-a$.

18. $\sqrt{x} + \sqrt{a - x} = a\sqrt{b}$.

Ans. $x = \frac{a \pm a\sqrt{2ab - a^2b^2}}{2}$.

19. $3x - \frac{14}{2x - 3} = 4x - 7$. *Ans.* $x = 5$ or $3\frac{1}{2}$.

20. $x - \frac{42}{x} = 1$. *Ans.* $x = 7$ or -6.

21. $2x + 1 - \frac{26}{2x + 1} = 11$. *Ans.* $x = 6$ or $-\frac{3}{2}$.

22. $\frac{23}{x + 2} - \frac{2}{x} = 3\frac{1}{3}$. *Ans.* $x = 4$ or $\frac{3}{10}$.

23. $(a + bx)(cx + d) = (a + b)(c + dx)$.

Ans. $x = 1$ or $\frac{ad - (a + b)c}{bc}$.

24. $(a - b)(x^2 - b^2) = (a + b)(x - b)^2$.

Ans. $x = b$ or a.

25. $(a + b)(x - a)(x - b) = abx$.

Ans. $x = a + b$ or $\frac{ab}{a + b}$.

26. $\frac{a}{x} + \frac{x}{b} = \frac{b}{x} + \frac{x}{a} + \frac{a^2 - b^2}{ab}$. *Ans.* $x = a$ or b.

27. $ax^2 - (c + d)x = bx^2 - \frac{cd}{a - b}$.

Ans. $x = \frac{c}{a - b}$ or $\frac{d}{a - b}$.

28. $a^3x^2 - (a-b)^2x + a^3 = b^3x^2 + (a^2+ab+b^2)^2x + b^3$.

Ans. $x = \frac{a^2 + ab + b^2}{a - b}$ or $\frac{a - b}{a^2 + ab + b^2}$.

262. To solve a complete quadratic equation by factoring.

EXERCISES.

1. Solve the equation $x^2 + 5x = 66$.

By transposition, $x^2 + 5x - 66 = 0$.

Factoring, $(x - 6)(x + 11) = 0$ **(100)**.

Now, if either of these factors be put equal to 0, the equation will be satisfied;

$\therefore$ $x - 6 = 0$; whence, $x = 6$;

or $x + 11 = 0$; whence, $x = -11$.

2. Solve the equation

$$x^2 - (a + b)x + ab = 0.$$

Factoring, $(x - a)(x - b) = 0$.

If either of these factors be put equal to 0, the equation will be satisfied;

$\therefore$ $x - a = 0$; whence, $x = a$;

or $x - b = 0$; whence, $x = b$.

This method of solving complete quadratics is very expeditious, when the roots are integral.

Solve, by factoring:

3. $x^2 - 5x + 6 = 0$. *Ans.* $x = 3$ or 2.
4. $x^2 - x = 72$. *Ans.* $x = 9$ or -8.
5. $x - \dfrac{42}{x} = 1$. *Ans.* $x = 7$ or -6.
6. $\dfrac{2x + 9}{x} + \dfrac{x}{2x + 9} = 5\frac{1}{5}$. *Ans.* $x = 3$ or -5.
7. $10x^2 + 40x = 50$. *Ans.* $x = 1$ or -5.
8. $9x^2 + 189x = 900$. *Ans.* $x = 4$ or -25.
9. $12x^2 - 12x = 24$. *Ans.* $x = 2$ or -1.
10. $x^2 + 6x - 16 = 0$. *Ans.* $x = 2$ or -8.
11. $x^2 + 10x - 75 = 0$. *Ans.* $x = 5$ or -15.

263. To solve, at sight, a complete quadratic equation of the standard form.

Solving the equation $x^2 + ax = b$ as in Art. **261**, the result is

$$x = -\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}.$$

Now $-\frac{a}{2}$ is half the *negative* of the coefficient of x, and $\sqrt{b + \frac{a^2}{4}}$ is the square root of the sum obtained by adding to the known term, b, the square of half the coefficient of x. Hence,

If a complete quadratic equation be reduced to the standard form ($x^2 + ax = b$), the value of x is equal to half the negative of the coefficient of the second term, plus or minus the square root of the sum obtained by adding to the known term the square of half the coefficient of the second term.

EXERCISES.

1. Solve the equation $\frac{x^2}{3} = 9 + \frac{x}{2}$.

Clearing of fractions, $2x^2 = 54 + 3x$.

By transposition, $2x^2 - 3x = 54$.

By division, $x^2 - \frac{3}{2}x = 27$.

The equation is now of the standard form.

The coefficient of x is $-\frac{3}{2}$, and the known term is 27; hence, by the theorem,

$$x = \frac{3}{4} \pm \sqrt{27 + \frac{9}{16}} = \frac{3}{4} \pm \sqrt{\frac{441}{16}} = \frac{3}{4} \pm \frac{21}{4}$$

$$= \frac{3 \pm 21}{4} = 6 \text{ or } -4\tfrac{1}{2}.$$

Solve:

2. $x^2 + 6x = 40$. *Ans.* $x = 4$ or -10.

3. $x^2 - 4x = 5$. *Ans.* $x = 5$ or -1.

4. $x^2 + 8x = 65$. *Ans.* $x = 5$ or -13.

5. $x^2 - 10x = 56$. *Ans.* $x = 14$ or -4.

6. $x^2 - 8x = 48$. *Ans.* $x = 12$ or -4.

7. $x^2 - 12x = 108$. *Ans.* $x = 18$ or -6.

8. $x^2 + 6x = 16$. *Ans.* $x = 2$ or -8.

9. $x^2 + 4x - 3 = 9$. *Ans.* $x = 2$ or -6.

10. $x^2 - 6x + 80 = 600$. *Ans.* $x = 26$ or -20.

11. $x^2 - 6x - 135 = 0$. *Ans.* $x = 15$ or -9.

12. $x^2 + x = 2$. *Ans.* $x = 1$ or -2.

13. $x^2 - x = 306$. *Ans.* $x = 18$ or -17.

14. $x^2 + 16x + 15 = 0$. *Ans.* $x = -1$ or -15.

15. $x^2 - 12x + 50 = 0$. *Ans.* $x = 6 \pm \sqrt{-14}$.

16. $x^2 + 5x = 24$. *Ans.* $x = 3$ or -8.

17. $x^2 + 11x = 12$. *Ans.* $x = 1$ or -12.

18. $x^2 - 17x = 84$. *Ans.* $x = 21$ or -4.

19. $x^2 + \frac{3}{4}x = \frac{85}{4}$. *Ans.* $x = 4\frac{1}{4}$ or -5.

20. $x^2 - \frac{5}{3}x = 14$. *Ans.* $x = 4\frac{2}{3}$ or -3.

21. $3x^2 - 30 = -27x$. *Ans.* $x = 1$ or -10.

22. $x^2 - 8ax = 3b^2$. *Ans.* $x = 4a \pm \sqrt{3b^2 + 16a^2}$.

23. $x^2 - 20x = a^2$. *Ans.* $x = 10 \pm \sqrt{100 + a^2}$.

24. $x^2 + \frac{2}{3}x + 14\frac{3}{4} = 16$. *Ans.* $x = \frac{5}{6}$ or $-1\frac{1}{2}$.

25. $6x^2 + 6 = 13x$. *Ans.* $x = \frac{3}{2}$ or $\frac{2}{3}$.

26. $x^2 - 5ax - c^2 = 0$. *Ans.* $x = \frac{5a \pm \sqrt{25a^2 + 4c^2}}{2}$.

27. $a - bx + cx^2 = 0.$ *Ans.* $x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2c}.$

28. $b(a - x)^2 = cx^2.$ *Ans.* $x = \dfrac{\pm a\sqrt{bc} - ab}{c - b}$

29. $(x + a)^2 + (x - a)^2 = 5ax.$ *Ans.* $x = 2a$ or $\dfrac{a}{2}.$

30. $a^2 - x^2 = b^2 - 2x.$ *Ans.* $x = 1 \pm \sqrt{a^2 - b^2 + 1}.$

31. $(a + b)x^2 - cx = \dfrac{ac}{a + b}.$

Ans. $x = \dfrac{c \pm \sqrt{4ac + c^2}}{2(a + b)}.$

32. $ax^2 - \dfrac{a^3 - b^2}{ab}x = 1.$ *Ans.* $x = \dfrac{a}{b}$ or $-\dfrac{b}{a^2}.$

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

264. 1. Find a number such that, if 36 be divided by it, the sum of the divisor and quotient shall be 15.

Let x = the number;

then $\dfrac{36}{x}$ = the quotient of 36 divided by the number.

Hence, by the problem,

$$\frac{36}{x} + x = 15;$$

whence, $36 + x^2 = 15x;$

whence, $x^2 - 15x = -36;$

whence, $x = \dfrac{15}{2} \pm \sqrt{\dfrac{225}{4} - 36}$

$= 12$ or 3 (**263**).

Both of these values satisfy the problem.

Verification. $\dfrac{36}{12} + 12 = 15;$ $\dfrac{36}{3} + 3 = 15.$

2. The sum of \$144 was divided equally among a certain number of persons. If the number of persons had been two less, each would have received one dollar more. Find the number of persons.

Let $x =$ the number of persons;

then $\frac{144}{x} =$ the number of dollars each received.

If there had been $x-2$ persons, each would have received $\frac{144}{x-2}$ dollars. Hence, by the problem,

$$\frac{144}{x-2} = \frac{144}{x} + 1;$$

whence, $144x = 144(x-2) + x(x-2);$

whence, $x^2 - 2x = 288;$

whence, $x = 18$ or -16.

Both values of x satisfy the *equation*, but the positive value is the only one that satisfies the *problem*. Hence, in solving problems by algebra, results are sometimes obtained which do not satisfy the problem actually proposed. The reason is that *the algebraic language* is more comprehensive than the language of the problem. Now, as 18 is the only value that satisfies this problem, what is the meaning of -16? It means that, if the words *more* and *less* in this problem be made to exchange places, the answer to the new problem will be 16. This will be seen by considering the following problem:

3. The sum of \$144 was divided equally among a certain number of persons. If the number of persons had been two *more*, each would have received one dollar *less*. Find the number of persons.

Let $x =$ the number of persons;

then $\frac{144}{x} =$ the number of dollars each received.

If there had been $x+2$ persons, each would have received $\frac{144}{x+2}$ dollars. Hence, by the problem,

$$\frac{144}{x+2}=\frac{144}{x}-1;$$

whence, $$x^2+2x=288;$$

whence, $$x=16 \text{ or } -18.$$

The roots of this equation are the *negatives* of those of the equation used in solving the preceding problem; and this is as it ought to be, for this equation may be derived from that by simply substituting $-x$ in the place of x. Thus,

$$\frac{144}{-x-2}=\frac{144}{-x}+1; \quad \text{whence,} \quad \frac{144}{x+2}=\frac{144}{x}-1.$$

4. Find two numbers, one of which is three times the other, and the sum of whose squares is 40.
Ans. ± 6 and ± 2.

5. Find two numbers, one of which is four times the other, and the difference of whose squares is 135.
Ans. ± 12 and ± 3.

6. Find a number whose square exceeds the number by 420. *Ans.* 21 or -20.

7. Find a number whose square is equal to the product of two other numbers, one of which is less by 6 than the required number, and the other greater by 9 than twice that number. *Ans.* 9 or -6.

8. Find a number whose square exceeds ten times the excess of the number above 2 by 95. *Ans.* 15.

9. Divide the number 50 into two such parts that their product shall be 369. *Ans.* 41 and 9.

10. Find a number such that, if it be subtracted from 18, and the remainder be multiplied by the number, the product shall be 65. *Ans.* 13 or 5.

11. A company dining at a hotel had to pay \$4; but before the bill was paid, three of the company went away; in consequence of which, each of those who remained had to pay 30 cents more than if all had been present. How many persons were in the company? *Ans.* 8.

12. A man traveled 30 miles at a uniform rate. If his rate per hour had been one mile less, his time would have been 5 hours greater. At what rate did he travel?

Ans. 3 miles per hour.

13. One hundred and ten bushels of meal were equally divided among a certain number of poor persons. If each person had received one bushel more, he would have received as many bushels as there were persons. Find the number of persons. *Ans.* 11.

14. A cistern can be supplied with water by two pipes; by one of them it would be filled 6 hours sooner than by the other, and by both together in 4 hours; find the time in which each pipe alone would fill it.

Ans. 6 hours, and 12 hours.

15. A and B started from the same place and at the same time, to go to a place distant 150 miles; A traveled 3 miles an hour faster than B, and arrived at the end of his journey 8 hours and 20 minutes before B: find the rate at which each traveled. *Ans.* 6 miles, and 9 miles, per hour.

16. A man bought some wine for \$300. If he had bought 3 dozen more for the same sum, he would have paid \$5 less per dozen. How many dozen did he buy?

Explain the negative answer. *Ans.* 12 or -15.

17. A and B distribute \$1200 each among a certain number of poor persons: A relieves 40 persons more than B, but B gives \$5 more than A to each person; find the number of persons relieved by A and B respectively.

Ans. 120 by A, and 80 by B.

18. Find two numbers whose difference is 9, and the difference of whose reciprocals is $\frac{1}{4}$. *Ans.* 12 and 3.

19. Find a number which exceeds its square root by 992.
Ans. 1024.

20. Find two numbers whose sum is 21, and the sum of whose squares is 225. *Ans.* 12 and 9.

21. Find two numbers whose sum is a, and the sum of whose squares is b. *Ans.* $\frac{a \pm \sqrt{2b - a^2}}{2}$ and $\frac{a \mp \sqrt{2b - a^2}}{2}$.

22. A man bought a certain quantity of coffee for $5.40. If the price of coffee were to rise 2½ cts. per lb., he would get 3 lbs. less for the same sum. How many lbs. of coffee did he buy? *Ans.* 27.

23. An express train started from New York an hour later than a freight train, which it overtook and passed on another line of rails 30 miles from New York; after 80 minutes longer, they were 36 miles apart: find their rates of traveling. *Ans.* 45 miles, and 18 miles.

24. A and B set out from different places to meet each other. They started at the same time, and traveled on the direct road between the two places. On meeting, it appeared that A had traveled 30 miles more than B; and that A could have traveled B's distance in 4 days, but that B would have been 9 days in traveling A's distance. Find the distance between the two places. *Ans.* 150 miles.

25. A and B set out from different places to meet each other. They started at the same time, and traveled on the direct road between the two places. On meeting, it appeared that A had traveled a miles more than B; and that A could have traveled B's distance in b days, but that B would have been c days in traveling A's distance. Find the distance between the two places. *Ans.* $a\frac{\sqrt{b} + \sqrt{c}}{\sqrt{c} - \sqrt{b}}$ miles.

Solve the 24th problem by substituting for a, b, c, in this result, their values.

THEORY OF QUADRATIC EQUATIONS.

265. TH. *The sum of the roots of a quadratic equation of the standard form is equal to the negative of the coefficient of the second term.*

The roots of the equation $x^2 + ax = b$ are

$$-\frac{a}{2} + \sqrt{b + \frac{a^2}{4}} \quad \text{and} \quad -\frac{a}{2} - \sqrt{b + \frac{a^2}{4}}.$$

The sum of these roots is $-a$, which is the negative of the coefficient of x in the given equation.

Find the sum of the roots of each of the following equations:

$$x^2 - 10x = -16, \qquad x^2 + x = 6,$$
$$x^2 + 10x = -16, \qquad x^2 - x = 210.$$

266. TH. *The product of the roots of a quadratic equation of the standard form is equal to the negative of the second member.* For

$$\left(-\frac{a}{2} + \sqrt{b + \frac{a^2}{4}}\right)\left(-\frac{a}{2} - \sqrt{b + \frac{a^2}{4}}\right) = \frac{a^2}{4} - \left(b + \frac{a^2}{4}\right)$$
$$= -b \ (\mathbf{83}).$$

Thus, the roots of the equation $x^2 - 10x = -16$ are 8 and 2, and their product is 16, which is the negative of the second member.

267. By means of the two preceding theorems, a quadratic equation can be found when its roots are given.

EXERCISES.

1. Find the equation whose roots are 3 and 5.

The first term of the equation is x^2, the coefficient of the second term is -8 (**265**), and the second member is -15 (**266**). Hence, $x^2 - 8x = -15$ is the equation required.

2. Find the equation whose roots are 2 and 3.
Ans. $x^2 - 5x = -6$.

3. Find the equation whose roots are -2 and -3.
Ans. $x^2 + 5x = -6$.

4. Find the equation whose roots are 5 and 0.
Ans. $x^2 - 5x = 0$.

5. Find the equation whose roots are $-\frac{1}{2}$ and $\frac{1}{3}$.
Ans. $x^2 + \frac{x}{6} = \frac{1}{6}$.

6. Find the equation whose roots are 3 and -3.
Ans. $x^2 = 9$.

7. Find the equation whose roots are $2 + \sqrt{-1}$ and $2 - \sqrt{-1}$.
Ans. $x^2 - 4x = -5$.

DISCUSSION OF THE ROOTS OF THE EQUATION

$$x^2 + ax = b.$$

268. TH. *If* a *and* b *are positive, the roots have contrary signs, and the negative root is numerically the greater.*

For, when b is positive, the product of the roots is negative (**266**); hence, they must have contrary signs. Again, when a is positive, the sum of the roots is negative (**265**); hence, the negative root must be numerically the greater.

Illustration. The roots of the equation $x^2 + 2x = 8$ are 2 and -4.

269. TH. *If* a *is negative, and* b *positive, the roots have contrary signs, and the positive root is numerically the greater.*

For, when b is positive, the product of the roots is negative; hence, they must have contrary signs. Again, when

a is negative, the sum of the roots is positive; hence, the positive root must be numerically the greater.

Illustration. The roots of the equation $x^2 - 2x = 8$ are 4 and -2.

270. Th. *If **a** is positive, and **b** negative, both roots are negative.*

For, when b is negative, the product of the roots is positive; hence, they have *like* signs. Again, when a is positive, the sum of the roots is negative; hence, both roots are negative.

Illustration. The roots of the equation $x^2 + 5x = -6$ are -2 and -3.

271. Th. *If **a** and **b** are negative, both roots are positive.*

For, when b is negative, the product of the roots is positive; hence, they have *like* signs. Again, when a is negative, the sum of the roots is positive; hence, both roots are positive.

Illustration. The roots of the equation $x^2 - 5x = -6$ are 2 and 3.

272. Th. *If **b** is positive, the roots are real.*

For, when b is positive, the radical part $\left(\sqrt{b + \frac{a^2}{4}}\right)$ of each root is the square root of a positive quantity, whatever the sign or the value of a may be.

273. Th. *If **b** is negative, the roots are real, when $\frac{a^2}{4}$ is not numerically less than **b**.*

For, when b is negative, and $\frac{a^2}{4}$ is not numerically less than b, the radical part of each root is the square root of a positive quantity.

274. TH. *If* b *is negative, the roots are imaginary, when* $\frac{a^2}{4}$ *is numerically less than* b.

For then the radical part of each root is the square root of a negative quantity (**213**), and the whole value is considered imaginary on account of the presence of the imaginary part.

Illustration. In the equation $x^2 - 6x = -10$ the square of half the coefficient of x is 9, which is numerically less than -10; hence, it is known in advance that the roots of the equation are imaginary.

This may be verified by solving the equation.

We find

$$x = 3 \pm \sqrt{-10 + 9} = 3 \pm \sqrt{-1}.$$

275. If a problem furnishes an equation whose roots are imaginary, the problem contains *incompatible conditions.* For example, let it be required to find two numbers whose sum shall be 10 and product 30.

Let $x =$ one of the numbers;

then $10 - x =$ the other.

Hence, by the problem,

$$(10 - x)\,x = 30;$$

whence, $10x - x^2 = 30;$

whence, $x^2 - 10x = -30;$

whence, $x = 5 \pm \sqrt{-30 + 25} = 5 \pm \sqrt{-5}.$

These imaginary roots indicate that there are no *real* numbers whose sum is 10 and product 30. The condition that the sum of the numbers shall be 10 is inconsistent with the other condition, that their product shall be 30.

HIGHER EQUATIONS.

CONTAINING ONLY ONE UNKNOWN QUANTITY.

276. Certain equations of a degree higher than the second may be solved in the same manner as quadratics.

Equations which can be solved in this way are reducible to three classes:

(1) Such as can be reduced to the form of $x^n = b$;

(2) Such as can be reduced to the form of $x^{2n} + ax^n = b$;

(3) Such as can be resolved, by factoring, into two or more equations, one of which, at least, is not of a higher degree than the second.

277. To solve an equation which is reducible to the form of $x^n = b$.

EXERCISES.

1. Solve the equation $x^3 = 8$.

Extracting the cube root of both members,

$$x = 2.$$

2. Solve the equation $x^4 = 16$.

Extracting the square root of both members,

$$x^2 = \pm 4;$$

whence, $$x = \pm 2 \text{ or } \pm \sqrt{-4}.$$

3. Solve the equation

$$x^3 - 6x^2 + 12x = 35.$$

Subtracting 8 from both members,

$$x^3 - 6x^2 + 12x - 8 = 27.$$

The first member of this equation is the cube of $x-2$;

$$\therefore \quad (x-2)^3 = 27.$$

Extracting the cube root of both members,

$$x - 2 = 3; \quad \text{whence,} \quad x = 5.$$

Solve

4. $x^3 + \dfrac{11 - x^3}{3} = \dfrac{26 - x^3}{2}$. *Ans.* $x = 2$

5. $19x^4 + \dfrac{7x^4 + \frac{1}{2}}{3} = 4x^4 + \dfrac{35}{2}$.
Ans. $x = \pm 1$ or $\pm\sqrt{-1}$.

6. $\dfrac{2x^5 + 1}{2} = \dfrac{7x^5 + 5}{8}$. *Ans.* $x = 1$.

7. $x^3 + 3x^2 + 3x = 7$. *Ans.* $x = 1$.

8. $x^3 - 9x^2 + 27x = 54$. *Ans.* $x = 6$.

278. To solve an equation which is reducible to the form of $x^{2n} + ax^n = b$.

EXERCISES.

1. Solve the equation $x^4 - 25x^2 = -144$.

Adding the square of half the coefficient of x^2 to both members,

$$x^4 - 25x^2 + \left(\frac{25}{2}\right)^2 = -144 + \frac{625}{4} = \frac{49}{4}.$$

Extracting the square root,

$$x^2 - \frac{25}{2} = \pm\frac{7}{2};$$

whence, $$x^2 = \frac{25}{2} \pm \frac{7}{2} = 16 \text{ or } 9;$$

whence, $$x = \pm 4 \text{ or } \pm 3.$$

2. Solve the equation

$$(x^2 + 3x)^2 + x^2 + 3x = 110. \qquad (1)$$

Assuming $x^2 + 3x = y$, the equation becomes

$$y^2 + y = 110; \qquad (2)$$

whence,

$$y = -\frac{1}{2} \pm \sqrt{110 + \frac{1}{4}} = -\frac{1}{2} \pm \frac{21}{2} = 10 \text{ or } -11.$$

$$\therefore \quad x^2 + 3x = 10, \qquad (3)$$

or

$$x^2 + 3x = -11. \qquad (4)$$

From (3),

$$x = -\frac{3}{2} \pm \sqrt{10 + \frac{9}{4}} = -\frac{3}{2} \pm \frac{7}{2} = 2 \text{ or } -5;$$

and from (4),

$$x = -\frac{3}{2} \pm \sqrt{-11 + \frac{9}{4}} = -\frac{3}{2} \pm \sqrt{-\frac{35}{4}}$$

$$= \frac{\pm\sqrt{-35} - 3}{2}.$$

3. Solve the equation $x^3 + 14x^{\frac{3}{2}} = 1107.$

Assuming $x^{\frac{3}{2}} = y$, this equation becomes

$$y^2 + 14y = 1107;$$

whence,

$$y = -7 \pm \sqrt{1107 + 49} = -7 \pm 34 = 27 \text{ or } -41;$$

$$\therefore \quad x^{\frac{3}{2}} = 27 \text{ or } -41;$$

whence, by extracting the cube root,

$$x^{\frac{1}{2}} = 3 \text{ or } (-41)^{\frac{1}{3}};$$

whence, by squaring, $\quad x = 9 \text{ or } (-41)^{\frac{2}{3}}.$

Solve:

4. $4x^4 - 11x^2 = 225.$

Ans. $x = \pm 3 \text{ or } \pm \frac{5}{2}\sqrt{-1}.$

5. $5x^6 - 17x^3 = 184$. *Ans.* $x = 2$ or $-\frac{1}{5}\sqrt[3]{575}$.

6. $x^2 + 5x^{-2} = 25\frac{1}{5}$. *Ans.* $x = \pm 5$ or $\pm \frac{\sqrt{5}}{5}$.

7. $3x^{-4} - 17x^{-2} = 1450$. *Ans.* $x = \pm \frac{1}{5}$.

8. $(x^2 + 3x + 3)^2 + 2x^2 = 189 - 6x$.
Ans. $x = 2$ or -5.

9. $\sqrt{x^2 + 9} + 20 = x^2 + 9$.
Ans. $x = \pm 4$ or $\pm \sqrt{7}$.

10. $\left(\frac{12}{x} - x\right)^2 + \frac{60}{x} = 6 + 5x$.
Ans. $x = 3, -4$, or $3 \pm \sqrt{21}$.

11. $x^2 - 7x + 8 + \sqrt{x^2 - 7x + 18} = 32$.
Ans. $x = 9, -2$, or $\frac{1}{2}\left(7 \pm \sqrt{173}\right)$.

12. $x^{10} - 31x^5 = 32$. *Ans.* $x = 2$ or -1.

13. $x^n + \frac{6}{x^n} = 5$. *Ans.* $x = \sqrt[n]{2}$ or $\sqrt[n]{3}$.

14. $x^3 - 6x^{\frac{3}{2}} = 567$. *Ans.* $x = 9$ or $\sqrt[3]{441}$.

15. $\sqrt{x + 1} - 2\sqrt[4]{x + 1} = 4$.
Ans. $x = \left(1 \pm \sqrt{5}\right)^4 - 1$.

16. $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11$.
Ans. $x = 1$ or $1 \pm 2\sqrt{15}$.

17. $x^n - 2ax^{\frac{n}{2}} = b^2$. *Ans.* $x = \left(a \pm \sqrt{a^2 + b^2}\right)^{\frac{2}{n}}$.

18. $(a + x)^{\frac{2}{3}} - 5\,(a^2 - x^2)^{\frac{1}{3}} = -4\,(a - x)^{\frac{2}{3}}$.
Ans. $x = \frac{63a}{65}$ or 0.

279. To solve an equation by factoring.

EXERCISES.

1. Solve the equation $x^3 - 5x^2 + 6x = 0$.

Factoring, $x(x^2 - 5x + 6) = 0$.

Now, if either of these factors be put equal to 0, the equation will be satisfied;

$$\therefore \quad x = 0,$$

or $x^2 - 5x + 6 = 0$; whence, $x = 3$ or 2.

The roots then are 0, 3, and 2.

2. Solve the equation $x^6 - 1 = 0$.

$$x^6 - 1 = (x^3 + 1)(x^3 - 1).$$

But $x^3 + 1 = (x^2 - x + 1)(x + 1)$,

and $x^3 - 1 = (x^2 + x + 1)(x - 1)$;

hence, the given equation may be reduced to

$$(x^2 - x + 1)(x + 1)(x^2 + x + 1)(x - 1) = 0.$$

Now, if any one of these factors be put equal to 0, the equation will be satisfied;

$$\therefore \quad x^2 - x + 1 = 0; \quad \text{whence,} \quad x = \frac{1 \pm \sqrt{-3}}{2};$$

$$x + 1 = 0; \quad \text{whence,} \quad x = -1;$$

$$x^2 + x + 1 = 0; \quad \text{whence,} \quad x = \frac{-1 \pm \sqrt{-3}}{2};$$

$$x - 1 = 0; \quad \text{whence,} \quad x = 1.$$

Solve

3. $x^3 - 4x^2 + 3x = 0$. *Ans.* $x = 0, 1,$ or 3.
4. $6x^4 - 13x^3 = -6x^2$. *Ans.* $x = 0, \frac{2}{3},$ or $\frac{3}{2}$.
5. $(x^2 - 4x + 3)(x^2 - 5x + 4) = 0$. *Ans.* $x = 1, 3,$ or 4.
6. $x(x - 1)(x - 2) = 6x$. *Ans.* $x = 0, 4,$ or -1.
7. $(x - 1)(x - 2)(x - 3)(x - 4) = 0$. *Ans.* $x = 1, 2, 3,$ or 4.

8. $x^3 - a^3 = 7a^2(x - a)$. *Ans.* $x = a,\ 2a$, or $-3a$.

9. $5x^2(a - x) = (a^2 - x^2)(x + 3a)$.

Ans. $x = a,\ \frac{3a}{2}$, or $-\frac{a}{2}$.

PROBLEMS LEADING TO HIGHER EQUATIONS.

280. 1. Some boys took 2916 apples from an orchard; and the number of apples that each took was equal to four times the square of the number of boys. Find the number of boys. *Ans.* 9.

2. Find two numbers such that, if the greater be multiplied by the square root of the less, the product shall be 48; but, if the less be multiplied by the square root of the greater, the product shall be 36. *Ans.* 16 and 9.

3. Find two numbers such that, if the less be multiplied by the square of the greater, the product shall be 18; but, if the greater be multiplied by the square of the less, the product shall be 12. *Ans.* 2 and 3.

4. Find a number such that its cube exceeds 4 as much as 68 exceeds the square root of the cube of that number. *Ans.* 4.

5. The difference between the cube root of the fourth power of a certain number and seven times the cube root of its square is 18. What is the number? *Ans.* ± 27.

6. Find two numbers whose sum is 5, and the sum of whose fourth powers is 97. *Ans.* 2 and 3.

7. A number n is diminished by the x^{th} part of itself; the remainder is diminished by the x^{th} part of itself; and so on to the fifth remainder, which is equal to r. Find the value of x.

Ans. $x = \frac{\sqrt[5]{n}}{\sqrt[5]{n} - \sqrt[5]{r}}$.

8. Find two numbers whose sum is 9, and the sum of whose fourth powers is 1377. *Ans.* 3 and 6.

SIMULTANEOUS EQUATIONS.

EXERCISES.

281. 1. Solve the equations

$$x + y = 9, \qquad (1)$$

$$x^2 + 2y^2 = 66. \qquad (2)$$

From (1), $y = 9 - x.$ (3)

Substituting $9 - x$ for y in (2),

$$x^2 + 2(9 - x)^2 = 66;$$

whence, $x = 8$ or 4; hence, by (3), $y = 1$ or 5.

This method may be used, whenever one of the given equations is of the first and the other of the second degree.

2. Solve the equations

$$2x^2 + 3y^2 = 35, \qquad (1)$$

$$3x^2 + 2y^2 = 30. \qquad (2)$$

Multiplying (1) by 2, and (2) by 3,

$$4x^2 + 6y^2 = 70, \qquad (3)$$

$$9x^2 + 6y^2 = 90. \qquad (4)$$

Subtracting (3) from (4), $5x^2 = 20$;

whence, $x = \pm 2$;

hence, by (2), $12 + 2y^2 = 30$;

whence, $y = \pm 3$.

There are then four pairs of values that satisfy the given equations:

(1) $x = 2, \quad y = 3$; (3) $x = -2, \quad y = 3$;

(2) $x = 2, \quad y = -3$; (4) $x = -2, \quad y = -3$.

3. Solve the equations

$$x^2 + xy = 35, \tag{1}$$

$$xy - y^2 = 6. \tag{2}$$

Assume $$y = vx; \tag{3}$$

then (1) and (2) become

$$x^2 + vx^2 = 35, \tag{4}$$

$$vx^2 - v^2x^2 = 6. \tag{5}$$

From (4), $$x^2 = \frac{35}{1+v}; \tag{6}$$

from (5), $$x^2 = \frac{6}{v - v^2}; \tag{7}$$

$$\therefore \quad \frac{35}{1+v} = \frac{6}{v - v^2};$$

whence, $$35v^2 - 29v = -6;$$

whence, $$v = \tfrac{3}{7} \text{ or } \tfrac{2}{5}.$$

Substituting $\frac{3}{7}$ for v in (3) and (6),

$$y = \tfrac{3}{7}x, \quad \text{and} \quad x^2 = \tfrac{49}{2};$$

whence, $x = \pm \frac{7}{2}\sqrt{2}$, and $y = \frac{3}{7}\left(\pm \frac{7}{2}\sqrt{2}\right) = \pm \frac{3}{2}\sqrt{2}$.

Substituting $\frac{2}{5}$ for v in the same equations,

$$y = \tfrac{2}{5}x, \qquad \text{and} \quad x^2 = 25;$$

whence, $x = \pm 5$, and $y = \pm 2$.

Hence, there are four pairs of values:

(1) $x = \frac{7}{2}\sqrt{2}$, $y = \frac{3}{2}\sqrt{2}$; (3) $x = 5$, $y = 2$;

(2) $x = -\frac{7}{2}\sqrt{2}$, $y = -\frac{3}{2}\sqrt{2}$; (4) $x = -5$, $y = -2$.

This method may be used, whenever the given equations are of the second degree and homogeneous (**46**).

4. Solve the equations

$$x^3 + y^3 = 9, \qquad (1)$$
$$x + y = 3. \qquad (2)$$

Dividing (1) by (2),

$$x^2 - xy + y^2 = 3. \qquad (3)$$

Squaring (2), $x^2 + 2xy + y^2 = 9.$ (4)

Subtracting (3) from (4), $3xy = 6;$

whence, $xy = 2.$ (5)

Subtracting (5) from (3),

$$x^2 - 2xy + y^2 = 1; \qquad (6)$$

whence, $x - y = \pm 1.$ (7)

Adding (2) and (7), $2x = 4$ or 2;

whence, $x = 2$ or 1;

$\therefore \quad y = 1$ or 2.

5. Solve the equations

$$x + y = 25, \qquad (1)$$
$$\sqrt{x} + \sqrt{y} = 7. \qquad (2)$$

Squaring (2), $x + 2\sqrt{xy} + y = 49.$ (3)

Subtracting (1) from (3), $2\sqrt{xy} = 24;$

whence, $4xy = 576.$ (4)

Squaring (1), $x^2 + 2xy + y^2 = 625.$ (5)

Subtracting (4) from (5),

$$x^2 - 2xy + y^2 = 49;$$

whence, $x - y = \pm 7.$ (6)

Adding (1) and (6), $2x = 32$ or 18;

whence, $x = 16$ or 9.

Subtracting (6) from (1), $2y = 18$ or 32;

whence, $y = 9$ or 16.

6. Solve the equations

$$(x+y)^3 + z^3 = 1125, \qquad (1)$$

$$x + y + z = 15, \qquad (2)$$

$$xy = 24. \qquad (3)$$

From (2), $$x + y = 15 - z; \qquad (4)$$

hence, by (1), $$(15 - z)^3 + z^3 = 1125;$$

whence, $$z^2 - 15z = -50;$$

whence, $$z = 10 \text{ or } 5.$$

Substituting 10 for z in (4),

$$x + y = 5. \qquad (5)$$

From (3), $$y = \frac{24}{x};$$

hence, by (5), $$x + \frac{24}{x} = 5;$$

whence, $$x = \frac{5 \pm \sqrt{-71}}{2};$$

hence, by (2), $$y = 15 - 10 - \frac{5 \pm \sqrt{-71}}{2} = \frac{5 \mp \sqrt{-71}}{2}.$$

Substituting 5 for z in (4),

$$x + y = 10;$$

hence, by (3), $$x + \frac{24}{x} = 10;$$

whence, $$x = 6 \text{ or } 4;$$

$$\therefore \quad y = 4 \text{ or } 6.$$

Solve:

7. $\left\{\begin{array}{l} x + y = 9 \\ x^2 + y^2 = 65 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 8 \text{ or } 1, \\ y = 1 \text{ or } 8. \end{array}\right.$

8. $\left\{\begin{array}{l} x + y = 9 \\ x^2 - y^2 = 63 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 8, \\ y = 1. \end{array}\right.$

9. $\left\{\begin{array}{r} x + y = 5 \\ xy = 6 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 3 \text{ or } 2, \\ y = 2 \text{ or } 3. \end{array}\right.$

10. $\left\{\begin{array}{r} x-y=2 \\ xy=15 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=5 \text{ or } -3, \\ y=3 \text{ or } -5. \end{array}\right.$

11. $\left\{\begin{array}{r} x-y=3 \\ x^2+y^2=29 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=5 \text{ or } -2, \\ y=2 \text{ or } -5. \end{array}\right.$

12. $\left\{\begin{array}{r} x-y=10 \\ x^2-y^2=520 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=31, \\ y=21. \end{array}\right.$

13. $\left\{\begin{array}{r} x+y=26 \\ x^2+xy+y^2=508 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=14 \text{ or } 12, \\ y=12 \text{ or } 14. \end{array}\right.$

14. $\left\{\begin{array}{r} x-y=6 \\ x^2+xy+y^2=777 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=19 \text{ or } -13, \\ y=13 \text{ or } -19. \end{array}\right.$

15. $\left\{\begin{array}{r} 2x+3y=12 \\ 5x^2-7y^2=17 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=3 \text{ or } -22\frac{13}{17}, \\ y=2 \text{ or } 19\frac{3}{17}. \end{array}\right.$

16. $\left\{\begin{array}{r} 3x-5y=24 \\ 2x^2-30y^2=68 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=13 \text{ or } 6\frac{7}{11}, \\ y=3 \text{ or } -\frac{9}{11}. \end{array}\right.$

17. $\left\{\begin{array}{r} x^2+y^2=650 \\ xy=323 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\pm 19, \\ y=\pm 17. \end{array}\right.$

18. $\left\{\begin{array}{r} x^2+y^2=613 \\ x^2-y^2=35 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\pm 18, \\ y=\pm 17. \end{array}\right.$

19. $\left\{\begin{array}{r} x^2-y^2=315 \\ xy=494 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\pm 26, \\ y=\pm 19. \end{array}\right.$

20. $\left\{\begin{array}{r} x^2+xy+y^2=607 \\ xy=69 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\pm 23, \\ y=\pm 3. \end{array}\right.$

21. $\left\{\begin{array}{r} x^2+3xy+y^2=1741 \\ xy=297 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\pm 27, \\ y=\pm 11. \end{array}\right.$

22. $\left\{\begin{array}{r} x^2-xy+y^2=1263 \\ 3xy=2706 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\pm 41, \\ y=\pm 22. \end{array}\right.$

23. $\left\{\begin{array}{r} x^2-xy=6 \\ x^2+y^2=61 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\pm 6 \text{ or } \pm\frac{1}{2}\sqrt{2}, \\ y=\pm 5 \text{ or } \mp\frac{11}{2}\sqrt{2}. \end{array}\right.$

24. $\left\{\begin{array}{r} x^2-xy=48 \\ xy-y^2=12 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\pm 8, \\ y=\pm 2. \end{array}\right.$

25. $\left\{\begin{array}{r}(x+2y)^2 = 256\\ 3y^2 - x^2 = 39\end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l}x = \pm 6 \text{ or } \pm 102,\\ y = \pm 5 \text{ or } \mp 59.\end{array}\right.$

26. $\left\{\begin{array}{r}(x+y)^2 + 2y^2 = 17\\ 2x^2 + 3xy + 5y^2 = 28\end{array}\right\}.$

Ans. $\left\{\begin{array}{l}x = \pm 1 \text{ or } \pm \frac{1}{2}\sqrt{2},\\ y = \pm 2 \text{ or } \pm \frac{3}{2}\sqrt{2}.\end{array}\right.$

27. $\left\{\begin{array}{r}x + y = 20\\ \sqrt{xy} = 8\end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l}x = 16 \text{ or } 4,\\ y = 4 \text{ or } 16.\end{array}\right.$

28. $\left\{\begin{array}{r}x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4\\ x^{\frac{3}{2}} + y^{\frac{3}{2}} = 28\end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l}x = 9 \text{ or } 1,\\ y = 1 \text{ or } 9.\end{array}\right.$

29. $\left\{\begin{array}{r}x + y = 35\\ x^{\frac{1}{3}} + y^{\frac{1}{3}} = 5\end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l}x = 27 \text{ or } 8,\\ y = 8 \text{ or } 27.\end{array}\right.$

30. $\left\{\begin{array}{r}xy^2z^3 = 108\\ yz^2 = 18x\\ 2z = 3x^2y\end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l}x = \pm 1,\\ y = \pm 2,\\ z = \pm 3.\end{array}\right.$

31. $\left\{\begin{array}{r}xyz = 105\\ 35x = 3yz\\ 7xy = 15z\end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l}x = \pm 3,\\ y = \pm 5,\\ z = \pm 7.\end{array}\right.$

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS.

282. 1. Find two numbers whose sum is 12, and product 20. *Ans.* 2 and 10.

2. The sum of two numbers is 13, and the sum of their squares is 125: find them. *Ans.* 2 and 11.

3. The difference of two numbers is 2, and the sum of their squares is 394: find them. *Ans.* 15 and 13.

4. The product of two numbers is 750, and their quotient is $3\frac{1}{3}$: find them. *Ans.* 50 and 15.

5. The sum of the squares of two numbers is 13001, and the difference of their squares is 1449: find them.

Ans. 85 and 76.

6. The sum of the squares of two numbers is 324900, and one of the numbers is $\frac{3}{4}$ of the other: find them.

Ans. 342 and 456.

7. Find two numbers such that their sum is 9, and the sum of their cubes, 243. *Ans.* 3 and 6.

8. Find two numbers such that their difference is 2, and the difference of their cubes, 602. *Ans.* 11 and 9.

9. Divide the number 16 into two parts, such that, if to their product the sum of their squares be added, the result shall be 208. *Ans.* 4 and 12.

10. The sum of two numbers is 20, and the square root of their product is 8: find them. *Ans.* 16 and 4.

11. Find two numbers, such that their sum, their product, and the difference of their squares shall be equal to each other. *Ans.* $\frac{3 \pm \sqrt{5}}{2}$ and $\frac{1 \pm \sqrt{5}}{2}$.

12. Find two numbers, such that three times their product exceeds the sum of their squares by 11, and twice their product exceeds the difference of their squares by 14.

Ans. 5 and 3.

13. If a certain number be divided by the product of its two digits, the quotient will be 6, and if 9 be added to the number, the order of the digits will be inverted. What is the number? *Ans.* 12.

14. The product of two numbers is 28, and if their sum be added to the sum of their squares, the result will be 76. What are the numbers? *Ans.* 4 and 7.

15. The sum of two numbers is 9, and the sum of their fourth powers is 2417. What are the numbers?

Ans. 2 and 7.

16. Divide the number 20 into two parts, such that the square of their product shall be 9216. *Ans.* 8 and 12.

17. A and B bought a farm of 1200 acres for $3600, each paying $1800. A paid $2.25 more per acre than B, in order

to be permitted to take his share from the best land. How many acres did each get, and at what price per acre?

Ans. A, 400 acres at \$4.50; B, 800 acres at \$2.25.

18. The fore-wheel of a wagon makes 6 revolutions more than the hind-wheel in passing over 120 meters; but, if the circumference of each wheel be increased by one meter, the fore-wheel will make only 4 revolutions more than the hind-wheel in the same distance. Find the circumference of each wheel. *Ans.* Fore-wheel, 4 meters; hind-wheel, 5 meters.

19. A man had \$2600 which he divided into two parts, and placed at interest at such rates that the incomes from them were equal. If he had put out the smaller part at the same rate as the larger, he would have drawn for this part \$72 interest in one year; but, if he had put out the larger part at the same rate as the smaller, the annual income from that part would have been \$98. Find the parts and the rates of interest.

Ans. \$1200 at 7 per cent., and \$1400 at 6 per cent.

20. Find three numbers, such that the sum of any two of them shall be equal to the reciprocal of the third.

Ans. Each is $\frac{1}{2}\sqrt{2}$.

21. A certain number consists of two digits. The left-hand digit is equal to 3 times the right-hand digit; and if 12 be subtracted from the number, the remainder will be equal to the square of the left-hand digit. What is the number? *Ans.* 93.

22. Two persons, A and B, leave Columbia, and walk in the same direction, at uniform rates. A starts 2 hours before B, and, after traveling 30 miles, B overtakes A. If each had traveled half a mile more per hour, B would have traveled 42 miles before overtaking A. Find their rates of walking. *Ans.* A $2\frac{1}{2}$ miles, B 3 miles, per hour.

CHAPTER XII.

RATIO AND PROPORTION.

RATIO.

283. The **Ratio** of two quantities is the quotient arising from dividing the first by the second. Thus, the ratio of 3 to 2 is $\frac{3}{2}$.

The ratio of two quantities, as a and b, may be expressed in any of the following ways:

$$\frac{a}{b}, \qquad a \div b, \qquad a : b.$$

284. The **Terms** of a ratio are the quantities compared. Thus, 3 and 2 are the terms of the ratio $3 : 2$.

285. The **Antecedent** of a ratio is its first term, and the **Consequent** is its second term.

The antecedent and consequent together form a *Couplet.*

286. The **Inverse**, or **Reciprocal Ratio** of two quantities is the ratio of their *reciprocals*, or the quotient arising from the division of the second by the first. Thus, the inverse ratio of a to b is $\frac{1}{a} : \frac{1}{b}$, or $\frac{b}{a}$.

287. TH. *The value of a ratio is not changed by multiplying or dividing both of its terms by the same quantity* **(128)**. Thus, the ratio of 3 to 2 is equal to the ratio of 3×5 to 2×5; that is, $\frac{3}{2} = \frac{3 \times 5}{2 \times 5}$.

EXERCISES.

288. 1. Which is the greater ratio, $12:6$ or $17:8$?

Ans. $17:8$.

2. Which is greater, $x^2+2 : x^2+4$ or $x^2+2 : x^2+5$.

Ans. $x^2+2 : x^2+4$.

3. What is the inverse ratio of d^2 to q? *Ans.* $\frac{q}{d^2}$.

4. Show that $m:n = 2m:2n$.

5. Show that $a^2 - x^2 : a^2 + x^2 > a - x : a + x$, if x is less than a.

6. Show that $m:n = \frac{m}{a} : \frac{n}{a}$.

7. Each of two vessels, A and B, contains a mixture of wine and water, A in the ratio of 3 to 2, and B in the ratio of 7 to 3. How many gallons must be drawn from each, in order to make a third mixture which shall contain 5 gallons of water and 11 gallons of wine?

Ans. 2 from A, and 14 from B.

8. There are two roads from A to B, one of them 14 miles longer than the other, and two roads from B to C, one of them 8 miles longer than the other. The distances from A to B and from B to C along the shorter roads are in the ratio of 1 to 2, and the distances along the longer roads are in the ratio of 2 to 3. Find the distances.

Ans. { From A to B, 26 miles and 40 miles;
" B to C, 52 miles and 60 miles.

9. Two numbers are in the ratio of 2 to 3, and if 9 be added to each of them, the results will be in the ratio of 3 to 4. What are the numbers? *Ans.* 18 and 27.

10. A grocer has two chests of mixed tea, in one of which the green and black are mixed in the ratio of 3 to 5, and in the other, in the ratio of 5 to 11. How many pounds must be taken from each chest, so as to form another mixture which shall consist of 19 lbs. of green tea and 37 lbs. of black? *Ans.* 24 lbs. from the 1st; 32 lbs. from the 2d.

PROPORTION.

289. A **Proportion** is an equation, each member of which is a ratio whose terms are expressed.

The equality of the ratios may be indicated either by the sign =, or by the double colon ::. Thus, the equality of the two ratios 3 : 2 and 15 : 10 may be indicated in any of the following ways:

$$3 : 2 = 15 : 10,$$

$$3 : 2 :: 15 : 10,$$

$$\frac{3}{2} = \frac{15}{10},$$

$$3 \div 2 = 15 \div 10.$$

This proportion, in any of its forms, may be read, "*The ratio of 3 to 2 is equal to the ratio of 15 to 10*, or, *3 is to 2 as 15 is to 10.*"

290. The **Extremes** of a proportion are the *first* and *fourth* terms; and

The **Means** are the *second* and *third* terms.

291. If $$a : b = c : d,$$

d is said to be a **Fourth Proportional** to a, b, and c.

292. If $$a : b = b : c,$$

c is said to be a **Third Proportional** to a and b, and b is said to be a **Mean Proportional** between a and c.

293. If four quantities, a, b, c, d, are so related that

$$a : b = \frac{1}{c} : \frac{1}{d},$$

they are said to be **Inversely, or Reciprocally Proportional.**

294. **A Continued Proportion** is a proportion which consists of more than two couplets. Thus,

$$1:3=2:6=5:15=4:12$$

is a continued proportion.

295. **Equimultiples** of two or more quantities are the products obtained by multiplying each of them by the same quantity. Thus, $3a$ and $3b$ are equimultiples of a and b.

296. TH. *The product of the extremes of a proportion is equal to the product of the means.*

Let $$a:b=c:d;$$

then $$\frac{a}{b}=\frac{c}{d}\quad \textbf{(289)}.$$

Multiplying both members by bd,

$$ad=bc.$$

COR. Let $$a:b=b:c;$$

then, by the theorem, $$b^2=ac;$$

whence, $$b=\sqrt{ac};$$ that is,

A mean proportional between two quantities is equal to the square root of their product.

297. TH. *If the product of two quantities is equal to the product of two others, either two may be made the extremes, and the other two the means of a proportion.*

Let $$ad=bc.$$

Dividing by bd, $$\frac{a}{b}=\frac{c}{d},$$

or $$a:b=c:d.$$

COR. 1. This theorem is a *test* of proportion, for, if the product of the first and fourth of four quantities is equal to the product of the second and third, they are in proportion by the theorem; but, if these products are unequal, the four quantities are not in proportion (**296**). Thus, $2:3=8:12$ is a proportion, because $2\times 12=3\times 8$; but $5:6=7:8$ is not a proportion, because $5\times 8<6\times 7$.

Hence, any change which does not affect the equality between the product of the extremes and the product of the means of a proportion is admissible.

COR. 2. Hence, (1) *If four quantities are in proportion, they will be in proportion by alternation;* that is, the means or the extremes may be made to exchange places. Thus, if $a:b=c:d$, then $a:c=b:d$ or $d:b=c:a$.

(2) *If four quantities are in proportion, they will be in proportion by inversion;* that is, the terms of each couplet may be made to exchange places. Thus, if $a:b=c:d$, then $b:a=d:c$.

(3) *Equimultiples of two quantities are in the same ratio as the quantities themselves.* Thus, $ma:mb=a:b$.

(4) One extreme and one mean may be multiplied or divided by the same quantity. Thus, if $a:b=c:d$, then $2a:b=2c:d$ or $\frac{a}{3}:b=\frac{c}{3}:d$.

(5) One extreme may be multiplied and the other divided by the same quantity. Thus, if $a:b=c:d$, then $2a:b=c:\frac{d}{2}$.

(6) One mean may be multiplied and the other divided by the same quantity. Thus, if $a:b=c:d$, then $a:2b=\frac{c}{2}:d$.

(7) All the terms may be multiplied or divided by the same quantity. Thus, if $a:b=c:d$, then $3a:3b=3c:3d$ or $\frac{a}{3}:\frac{b}{3}=\frac{c}{3}:\frac{d}{3}$.

(8) Both terms of one couplet may be multiplied or divided by one quantity, and both terms of the other couplet may be multiplied or divided by another quantity. Thus, if $a : b = c : d$, then $ma : mb = nc : nd$ or $\frac{a}{m} : \frac{b}{m} = \frac{c}{n} : \frac{d}{n}$.

298. Th. *If four quantities are in proportion, they will be in proportion by composition.*

The word *composition*, as applied to proportions, is used to express the *addition* of the antecedent and consequent in each ratio.

Let $$a : b = c : d; \qquad (1)$$

then $$\frac{a}{b} = \frac{c}{d}; \qquad (2)$$

whence, $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}; \qquad (3)$$

that is, $$a + b : b = c + d : d.$$

Dividing (3) by (2), $$\frac{a+b}{a} = \frac{c+d}{c};$$

that is, $$a + b : a = c + d : c.$$

299. Th. *If four quantities are in proportion, they will be in proportion by division.*

The word *division*, as applied to proportions, is used to express the *subtraction* of consequent from antecedent, or antecedent from consequent, in each ratio.

Let $$a : b = c : d; \qquad (1)$$

then $$\frac{a}{b} = \frac{c}{d}. \qquad (2)$$

whence, $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or
$$\frac{a-b}{b}=\frac{c-d}{d}; \qquad (3)$$

that is, $a-b:b=c-d:d.$

Dividing (3) by (2), $\dfrac{a-b}{a}=\dfrac{c-d}{c};$

that is, $a-b:a=c-d:c.$

Subtracting each member of (2) from 1,

$$1-\frac{a}{b}=1-\frac{c}{d};$$

whence,
$$\frac{b-a}{b}=\frac{d-c}{d};$$

that is, $b-a:b=d-c:d.$

Cor. Dividing (3) of Art. **298** by (3) of this Article,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d};$$

that is, $a+b:a-b=c+d:c-d;$

hence,

If four quantities are in proportion, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

300. Th. *The sum of any number of the antecedents of a continued proportion is to the sum of the corresponding consequents as any antecedent is to its consequent.*

Let $a:b=c:d=e:f=g:h=$ etc.,

and let r denote the ratio;

then $r=\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\dfrac{g}{h}=$ etc.;

whence, $br=a, \quad dr=c, \quad fr=e, \quad hr=g,$ etc.;

whence,

$$br + dr + fr + hr + \text{ etc.} = a + c + e + g + \text{ etc.};$$

whence;

$$(b + d + f + h + \text{ etc.})r = a + c + e + g + \text{ etc.};$$

whence,

$$\frac{a + c + e + g + \text{ etc.}}{b + d + f + h + \text{ etc.}} = r = \frac{a}{b} = \frac{c}{d} = \text{ etc.};$$

that is,

$$a+c+e+g+ \text{ etc.} : b+d+f+h+ \text{ etc.} = a : b = \frac{c}{d} = \text{ etc.}$$

301. TH. *The products of the corresponding terms of two or more proportions are in proportion.*

Let $a : b = c : d,$

$e : f = g : h,$

and $m : n = p : q;$

then $\frac{a}{b} = \frac{c}{d},$

$\frac{e}{f} = \frac{g}{h},$

and $\frac{m}{n} = \frac{p}{q};$

whence, $\frac{a}{b} \times \frac{e}{f} \times \frac{m}{n} = \frac{c}{d} \times \frac{g}{h} \times \frac{p}{q};$

whence, $\frac{aem}{bfn} = \frac{cgp}{dhq};$

that is, $aem : bfn = cgp : dhq.$

302. TH. *The quotients of the corresponding terms of two proportions are in proportion.*

Let $a : b = c : d,$

and $e : f = g : h;$

then $ad = bc,$

and $eh = fg;$

whence, $\dfrac{ad}{eh} = \dfrac{bc}{fg};$

whence, $\dfrac{a}{e} \times \dfrac{d}{h} = \dfrac{b}{f} \times \dfrac{c}{g};$

whence, $\dfrac{a}{e} : \dfrac{b}{f} = \dfrac{c}{g} : \dfrac{d}{h}$ **(297)**.

303. Th. *Like powers or like roots of the terms of a proportion are in proportion.*

Let $a : b = c : d;$

then $\dfrac{a}{b} = \dfrac{c}{d};$

whence, $\dfrac{a^n}{b^n} = \dfrac{c^n}{d^n},$ and $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \dfrac{\sqrt[n]{c}}{\sqrt[n]{d}}$ (**64**, 4, 5);

that is, $a^n : b^n = c^n : d^n,$ and $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$

EXERCISES.

304. 1. Find the fourth proportional to a, b, and c.

Ans. $\dfrac{bc}{a}$.

2. Find the third proportional to m and n. *Ans.* $\dfrac{n^2}{m}$.

3. Find the mean proportional between m^2 and n^2.

Ans. mn.

4. Solve $\left\{\begin{array}{l} x + y : 7 = x : 5 \\ y(x + y) = 126 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = \pm 15, \\ y = \pm 6. \end{array}\right.$

5. Solve $\left\{\begin{array}{l}\sqrt{x} \quad\; - \sqrt{x-y} = \sqrt{20-y} \\ \sqrt{x-y} : \sqrt{20-y} = 3 : 2\end{array}\right\}$.

Ans. $\left\{\begin{array}{l}x = 25, \\ y = 16.\end{array}\right.$

6. Solve $\left\{\begin{array}{r}x^2 + y^2 : a = 2a : 1 \\ x + y : x - y = m : n\end{array}\right\}$.

Ans. $\left\{\begin{array}{l}x = \pm \dfrac{a(m+n)}{\sqrt{m^2+n^2}}, \\ y = \pm \dfrac{a(m-n)}{\sqrt{m^2+n^2}}.\end{array}\right.$

7. If $\frac{a}{b} = \frac{c}{d}$, and $\frac{c}{e} = \frac{e}{d}$, show that $\frac{a}{b} = \frac{c^2}{e^2}$.

8. Divide 100 into two parts, such that their product shall be to the sum of their squares as 4 is to 17.

Ans. 80 and 20.

9. Divide 28 into two parts, such that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 16 is to 9. *Ans.* 16 and 12.

10. Find a number, such that, if it be added to a, and subtracted from b, the results shall be in the ratio of m to n.

Ans. $\frac{bm - an}{m + n}$.

11. Show that $a \pm \frac{m}{n} a : b \pm \frac{m}{n} b = a : b$, and state the theorem which the formula expresses.

12. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a^3 + c^3}{b^3 + d^3} = \left(\frac{a + c}{b + d}\right)^3$.

13. Find two numbers, the less of which shall be to the greater as 21 is to their sum, and as 3 is to their difference.

Ans. 12 and 16.

14. The product of two numbers is 112, and the difference of their squares is to the square of their difference as 11 is to 3. What are the numbers? *Ans.* 14 and 8.

CHAPTER XIII.

PROGRESSIONS.

ARITHMETICAL PROGRESSION.

305. An **Arithmetical Progression** is a collection of quantities arranged in the order of their magnitudes, each of which is greater or less than the preceding one by a constant difference. Thus, 1, 4, 7, 10, 13 is an arithmetical progression.

An arithmetical progression is sometimes called a *Progression by Difference.*

306. The **Terms** of an A. P. are the quantities of which the progression consists.

307. The **Extremes** of an A. P. are the first and the last terms; the other terms are the **Means.**

308. The **Common Difference** of an A. P. is the result obtained by subtracting any term from the one which immediately follows it. Thus, in the progression 1, 4, 7, 10, 13, the common difference is 3.

309. An **Increasing A. P.** is a progression in which the common difference is *positive.* Thus, 1, 3, 5, 7, 9 is an increasing A. P.

310. A **Decreasing A. P.** is a progression in which the common difference is *negative.* Thus, 9, 7, 5, 3, 1 is a decreasing A. P.

311. To find the n^{th} term and the sum of n terms of an A. P.

Let a denote the first term, d the common difference, l the n^{th} term, and s the sum of n terms; then it is evident that the progression will be

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+4d, \quad \text{and so on,}$$

the coefficient of d in any term being always less by 1 than the number of the term. Thus, in the 4th term, the coefficient of d is 3. Hence, the n^{th} term

$$l = a + (n-1)\,d. \tag{1}$$

Again,

$$s = a+(a+d)+(a+2d)+(a+3d)+\dots\dots+(l-d)+l;$$

or, reversing the order of the terms,

$$s = l+(l-d)+(l-2d)+(l-3d)+\dots\dots+(a+d)+a.$$

Adding these two equations,

$$2s = (a+l) + (a+l) + (a+l) + (a+l) + \dots\dots + (a+l) + (a+l);$$

and since there are n terms, each of which is $a+l$, in the second member of this equation,

$$2s = n\,(a+l);$$

whence,

$$s = \frac{n}{2}(a+l). \tag{2}$$

EXERCISES.

1. The first term of an A. P. is 5, the common difference 3, and the number of terms 24. Find the last term and the sum of the terms.

By (1), $l = 5 + (24-1)\,3 = 74.$

By (2), $s = \frac{24}{2}(5+74) = 948.$

Find the last term and the sum of the terms of

2. 1, 3, 5, ... to 7 terms. *Ans.* $l = 13$, $s = 49$.
3. 2, 5, 8, ... to 11 terms. *Ans.* $l = 32$, $s = 187$.
4. -1, 1, 3, ... to 6 terms. *Ans.* $l = 9$, $s = 24$.
5. 16, 13, 10, ... to 7 terms. *Ans.* $l = -2$, $s = 49$.
6. $a+3b$, $a+5b$, $a+7b$, ... to 6 terms.
   Ans. $l = a + 13b$, $s = 6a+48b$.

312. TH. *In an A. P. the sum of any two terms equidistant from the extremes is equal to the sum of the extremes.*

Let x represent the m^{th} term from the beginning, and y the m^{th} term from the end of the progression; then

$$x = a + (m - 1)d,$$

and
$$y = l - (m - 1)d;$$

$$\therefore\ x + y = a + l.$$

313. To insert any number of arithmetical means between two given quantities.

Let it be required to insert m arithmetical means between b and c.

Since there are to be m means, the number of terms will be $m+2$. Substituting b for a, c for l, and $m+2$ for n, in the formula $l = a + (n - 1)d$,

$$c = b + (m + 2 - 1)d;$$

whence,
$$d = \frac{c - b}{m + 1}.$$

The second term may now be found by adding the common difference to b, the third by adding it to the second, and so on.

EXERCISES.

1. Insert 8 arithmetical means between 5 and 32.

$$d = \frac{32 - 5}{8 + 1} = 3;$$

hence, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32 is the progression required.

2. Insert 2 arithmetical means between -3 and 3.

Ans. $-3, -1, 1, 3$.

3. Insert 7 arithmetical means between 1 and $-\frac{1}{2}$.

Ans. $1, \frac{13}{16}, \frac{5}{8}, \frac{7}{16}, \frac{1}{4}, \frac{1}{16}, -\frac{1}{8}, -\frac{5}{16}, -\frac{1}{2}$.

4. Insert one arithmetical mean between 1 and 5.

Ans. 1, 3, 5.

5. Insert 9 arithmetical means between 1 and -1.

Ans. $1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, -\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}, -\frac{4}{5}, -1$.

314. To find any two of the quantities a, l, d, n, and s, when the others are given.

The equations $$l = a + (n - 1)\,d, \tag{1}$$

and $$s = \frac{n}{2}(a + l) \tag{2}$$

contain these quantities; hence, the values of any two of them can be found when the others are given. Each case is an exercise in the solution of a pair of simultaneous equations containing only two unknown quantities.

EXERCISES.

1. Given a, d, and n, to find l and s.

Ans. $l = a + (n - 1)\,d,\ \ s = \frac{n}{2}[2a + (n - 1)\,d]$.

The value of l is found by (1), and the value of s, by substituting that value for l in (2).

2. Given $a = 5$, $d = 2$, and $n = 25$, to find l and s.

Ans. $l = 53$, $s = 725$.

Substitute the given values in the formulas in the answer to Ex. 1.

3. Given a, d, and l, to find n and s.

$$\textit{Ans. } n = \frac{l-a}{d} + 1, \quad s = \frac{(l+a)(l-a+d)}{2d}.$$

4. Given $a = 5$, $d = 3$, and $l = 74$, to find n and s.

Ans. $n = 24$, $s = 948$.

5. Given a, d, and s, to find n and l.

$$\textit{Ans. } \begin{cases} n = \dfrac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}, \\ l = \dfrac{-d \pm \sqrt{(2a-d)^2 + 8ds}}{2}. \end{cases}$$

6. Given $a = 15$, $d = -2$, and $s = 60$, to find n and l.

Ans. $n = 6$ or 10, $l = 5$ or -3.

7. Given a, n, and l, to find d and s.

$$\textit{Ans. } d = \frac{l-a}{n-1}, \quad s = \frac{n}{2}(a+l).$$

8. Given $a = 15$, $n = 6$, and $l = 5$, to find d and s.

Ans. $d = -2$, $s = 60$.

9. Given a, n, and s, to find d and l.

$$\textit{Ans. } d = \frac{2(s-an)}{n(n-1)}, \quad l = \frac{2s}{n} - a.$$

10. Given $a = 5$, $n = 6$, and $s = 60$, to find d and l.

Ans. $d = 2$, $l = 15$.

11. Given a, l, and s, to find d and n.

$$\textit{Ans. } d = \frac{l^2 - a^2}{2s - a - l}, \quad n = \frac{2s}{a+l}.$$

12. Given $a = -3$, $l = 15$, and $s = 60$, to find d and n.

Ans. $d = 2$, $n = 10$.

13. Given d, n, and l, to find a and s.

Ans. $a = l - (n-1)d$, $s = \frac{n}{2}[2l - (n-1)d]$.

14. Given $d = 5$, $n = 8$, and $l = 37$, to find a and s.

Ans. $a = 2$, $s = 156$.

15. Given d, n, and s, to find a and l.

Ans. $a = \frac{2s - n(n-1)d}{2n}$, $l = \frac{2s + n(n-1)d}{2n}$.

16. Given $d = 5$, $n = 8$, and $s = 156$, to find a and l.

Ans. $a = 2$, $l = 37$.

17. Given d, l, and s, to find a and n.

$$\textit{Ans.}\ \begin{cases} a = \dfrac{d \pm \sqrt{(2l+d)^2 - 8ds}}{2}, \\[2ex] n = \dfrac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}. \end{cases}$$

18. Given $d = 2$, $l = 15$, and $s = 60$, to find a and n.

Ans. $a = 5$ or -3, $n = 6$ or 10.

19. Given l, n, and s, to find a and d.

Ans. $a = \frac{2s}{n} - l$, $d = \frac{2(ln - s)}{n(n-1)}$.

20. Given $l = 5$, $n = 46$, and $s = 6440$, to find a and d.

Ans. $a = 275$, $d = -6$.

21. Find the sum of the first n terms of the progression 1, 2, 3, 4, 5, 6, ...

Ans. $s = \frac{n}{2}(1 + n)$.

22. Find the n^{th} term and the sum of the first n terms of the progression 1, 3, 5, 7, 9, ...

Ans. n^{th} term $= 2n - 1$, $s = n^2$.

23. If a man travels 26 miles the first day, 24 miles the second, 22 miles the third, and so on, how long will it take him to travel 140 miles?

Ans. 7 days.

By the formula, $n = 7$ or 20. How is the second value to be explained?

24. Write an A. P. of 6 terms whose extremes shall be 37 and 7. *Ans.* 37, 31, 25, 19, 13, 7.

25. A sets out from a place and travels $2\frac{1}{2}$ miles an hour. Three hours later, B sets out from the same place, and travels in the same direction, 3 miles the first hour, $3\frac{1}{2}$ miles the second, 4 miles the third, and so on. In how many hours will B overtake A? *Ans.* 5.

26. The seventh term of an A. P. is 12, the twelfth term is 7, and the sum of the terms is 171. Find the number of terms and the common difference.

Ans. $n = 19$ or 18, $d = -1$.

ARITHMETICAL MEAN.

315. The **Arithmetical Mean** of two or more quantities is the quotient obtained by dividing their sum by their number. Thus, the arithmetical mean of 2, 7, 10, and 5 is 6.

316. To find the arithmetical mean of the terms of an A. P.

Denote the arithmetical mean by m; then, by definition,

$$m = \frac{s}{n} = \frac{a+l}{2} \quad \textbf{(311)}.$$

317. To find a and l, when m, d, and n are given.

It has been shown that

$$a = \frac{s}{n} - \frac{(n-1)d}{2},$$

and

$$l = \frac{s}{n} + \frac{(n-1)d}{2} \quad (\textbf{314}, 15);$$

$$\therefore \quad a = m - \frac{(n-1)d}{2},$$

and

$$l = m + \frac{(n-1)d}{2}.$$

318. The arithmetical mean may be used to advantage in the solution of certain problems in arithmetical progression.

EXERCISES.

1. Find four numbers in A. P. whose sum is 16, and the sum of whose squares is 84.

Denote the common difference by x; then, since

$$m = \frac{16}{4} = 4,$$

$$\text{the first term} \quad = 4 - \frac{3}{2}x \ \textbf{(317)};$$

$$\text{the second term} \quad = 4 - \frac{3}{2}x + x = 4 - \frac{x}{2};$$

$$\text{the third term} \quad = 4 + \frac{x}{2};$$

$$\text{the fourth term} \quad = 4 + \frac{3}{2}x;$$

$$\therefore \quad \left(4 - \frac{3}{2}x\right)^2 + \left(4 - \frac{x}{2}\right)^2 + \left(4 + \frac{x}{2}\right)^2 + \left(4 + \frac{3}{2}x\right)^2 = 84;$$

whence, $$64 + 5x^2 = 84;$$

whence, $$x = \pm 2.$$

Hence, the required numbers are 1, 3, 5, 7.

2. Find three numbers in A. P., the sum of whose squares is 308, and the square of whose arithmetical mean exceeds the product of the extremes by 4. *Ans.* 8, 10, 12.

3. Find four numbers in A. P., whose sum is 16, and whose continued product is 105. *Ans.* 1, 3, 5, 7.

4. The arithmetical mean of seven numbers in A. P. is 6, and the product of the extremes is 27. What are the numbers? *Ans.* 3, 4, 5, 6, 7, 8, 9.

5. The sum of the squares of the extremes of an A. P. of four terms is 800, and the sum of the squares of the means is 544. What is the progression? *Ans.* 4, 12, 20, 28.

6. Find five numbers in A. P., whose arithmetical mean is 5, and continued product 945.

$$\textit{Ans.}\left\{\begin{array}{l} 1,\ 3,\ 5,\ 7,\ 9;\ \text{ or} \\ 5-\sqrt{109},\ 5-\frac{1}{2}\sqrt{109},\ 5,\ 5+\frac{1}{2}\sqrt{109},\ 5+\sqrt{109}. \end{array}\right.$$

7. The arithmetical mean of an A. P. of 18 terms is $15\frac{3}{4}$, and the product of the extremes is $85\frac{1}{2}$. Find the first term and the common difference. *Ans.* $a = 3$; $d = 1\frac{1}{2}$.

GEOMETRICAL PROGRESSION.

319. A **Geometrical Progression** is a collection of quantities arranged in the order of their magnitudes, each of which is greater or less than the preceding one in a constant ratio. Thus, 1, 3, 9, 27, 81 is a geometrical progression.

320. The **Terms** of a G. P. are the quantities of which the progression consists.

321. The **Extremes** of a G. P. are the first and the last terms; the other terms are the **Means.**

322. The **Ratio** of a G. P. is the quotient obtained by dividing any term by the one which immediately precedes it. Thus, in the progression 1, 3, 9, 27, the ratio is 3.

323. An **Increasing G. P.** is a progression in which the ratio is greater than 1. Thus, 2, 4, 8, 16, 32 is an increasing G. P.

324. A **Decreasing G. P.** is a progression in which the ratio is less than 1. Thus, 81, 27, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$ is a decreasing G. P.

325. An **Infinite Decreasing G. P.** is a progression in which the ratio is less than 1, and the number of terms is infinite.

326. To find the n^{th} term and the sum of n terms of a G. P.

Let a denote the first term, r the ratio, l the n^{th} term, and s the sum of n terms; then the progression will be

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \text{and so on,}$$

the exponent of r in any term being always less by 1 than the number of the term. Hence, the n^{th} term

$$l = ar^{n-1}. \tag{1}$$

Again, $s = a + ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-2} + ar^{n-1}$;

whence, $rs = \quad ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-2} + ar^{n-1} + ar^n$.

By subtraction, $\quad rs - s = ar^n - a$;

whence, $$s = a\frac{r^n - 1}{r - 1}. \tag{2}$$

Multiplying both members of (1) by r,

$$lr = ar^n;$$

hence, (2) may be reduced to

$$s = \frac{lr - a}{r - 1}. \tag{3}$$

EXERCISES.

1. The first term of a G. P. is 2, the ratio 3, and the number of terms 5. Find the last term and the sum of the terms.

By (1), $\quad l = 2 \times 3^4 = 162.$

By (2), $\quad s = 2 \times \dfrac{3^5 - 1}{3 - 1} = 242.$

Find the last term and the sum of the terms of

2. $1, 3, 9, \ldots$ to 5 terms. *Ans.* $l = 81$, $s = 121$.
3. $2, 4, 8, \ldots$ to 8 terms. *Ans.* $l = 256$, $s = 510$.
4. $3, \frac{3}{2}, \frac{3}{4}, \ldots$ to 7 terms. *Ans.* $l = \frac{3}{64}$, $s = 5\frac{61}{64}$.
5. $5, \frac{5}{3}, \frac{5}{9}, \ldots$ to 4 terms. *Ans.* $l = \frac{5}{27}$, $s = 7\frac{11}{27}$.
6. $3, -\frac{1}{2}, \frac{1}{12}, \ldots$ to 4 terms. *Ans.* $l = -\frac{1}{72}$, $s = 2\frac{41}{72}$.
7. $1, \frac{1}{2}, \frac{1}{4}, \ldots$ to 10 terms. *Ans.* $l = \frac{1}{512}$, $s = 1\frac{511}{512}$.

327. To find the limit of the sum of the terms of an infinite decreasing G. P.

Equation (2) of Art. 326 may be reduced to

$$s = \frac{a - ar^n}{1 - r} \quad \text{(127, Cor.)}.$$

Now, if $r < 1$, r^n becomes smaller, or approaches 0, as n increases. Thus, if $r = \frac{1}{2}$, and $n = 2, 3, 4, 5, 6$, etc., then $r^n = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, etc. Hence, the greater the value of n, the nearer does the sum of the terms approach the value of the fraction $\frac{a}{1 - r}$; that is, when $r < 1$,

$$\textit{Limit} \text{ of } s = \frac{a}{1 - r}.$$

EXERCISES.

1. Find the value of the recurring decimal, .666 etc.

The value required is the limit of the sum of $\frac{6}{10}, \frac{6}{100}, \frac{6}{1000}$, etc., to infinity. By the formula,

$$\text{limit of } s = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{2}{3}.$$

The meaning of this result is that $\frac{2}{3}$ is the fraction to which the value of .666 etc. approaches as the number of decimal figures is increased.

Find the limit of

2. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \text{etc.}$ *Ans.* 2.
3. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \text{etc.}$ *Ans.* $\frac{3}{5}$.
4. $10 - 9 + 8.1 - 7.29 + \text{etc.}$ *Ans.* $5\frac{5}{19}$.
5. $5\frac{1}{3} + 1\frac{1}{3} + \frac{1}{3} + \frac{1}{12} + \text{etc.}$ *Ans.* $7\frac{1}{9}$.
6. $.6 - .06 + .006 - \text{etc.}$ *Ans.* $\frac{6}{11}$.

328. TH. *In a G. P., the product of any two terms equidistant from the extremes is equal to the product of the extremes.*

Let x represent the m^{th} term from the beginning, and y the m^{th} term from the end of the progression; then

$$x = ar^{m-1},$$

and $$y = l\left(\frac{1}{r}\right)^{m-1};$$

$$\therefore \quad xy = al.$$

329. To insert any number of geometric means between two given quantities.

Let it be required to insert m geometric means between b and c.

Substituting b for a, c for l, and $m+2$ for n, in the formula $l = ar^{n-1}$,

$$c = br^{m+1};$$

whence, $$r^{m+1} = \frac{c}{b};$$

whence, $$r = \sqrt[m+1]{\frac{c}{b}}.$$

The second term may now be found by multiplying the first term by the ratio, the third, by multiplying the second by the ratio, and so on.

EXERCISES.

1. Insert 3 geometric means between 1 and 81.

$$r = \sqrt[4]{\frac{81}{1}} = 3;$$

hence, 1, 3, 9, 27, 81, is the progression required.

2. Insert 2 geometric means between 2 and 16.

Ans. 2, 4, 8, 16.

3. Insert 3 geometric means between $\frac{1}{2}$ and $\frac{1}{32}$.

Ans. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$.

4. Insert 5 geometric means between .6 and .0000006.

Ans. .6, .06, .006, .0006, .00006, .000006, .0000006.

5. Insert one geometric mean between 9 and 81.

Ans. 9, 27, 81.

6. Insert 2 geometric means between 1 and $-\frac{8}{27}$.

Ans. $1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}$.

330. To find the continued product of the terms of a G. P.

Denote the required product by P; then

$$P = a \times ar \times ar^2 \times ar^3 \times ar^4 \times \dots\dots \times ar^{n-1}$$
$$= a^n \times r^{1+2+3+4+\dots\dots+n-1}$$
$$= a^n \times r^{\frac{n-1}{2}n} \quad \textbf{(311)};$$

whence, $P^2 = a^{2n} \times r^{(n-1)n} = a^n \times a^n r^{(n-1)n} = a^n \times (ar^{n-1})^n$
$= a^n l^n$ **(326)**;

whence, $P = \sqrt{(al)^n}$.

Thus, the continued product of the terms of a G. P. of five terms, whose extremes are 2 and 32 is

$$\sqrt{(2 \times 32)^5} = \sqrt{(64)^5} = 8^5 = 32768.$$

331. To find any two of the quantities a, l, r, n, and s, when the others are given.

The equations $l = ar^{n-1}$, (1)

and $s = \dfrac{ar^n - 1}{r - 1}$ (2)

contain these quantities; hence, the values of any two of them can be found when the others are given.

EXERCISES.

1. Given a, r, and n, to find l and s.

$$Ans.\ l = ar^{n-1},\ s = a\frac{r^n - 1}{r - 1}.$$

These values are given by (1) and (2).

2. Given l, r, and n to find a and s.

$$Ans.\ a = \frac{l}{r^{n-1}},\ s = \frac{l(r^n - 1)}{r^n - r^{n-1}}.$$

Find a from (1); then substitute its value for a in (2).

3. Given $l = 6144$, $r = 2$, and $n = 12$, to find a and s.

$$Ans.\ a = 3,\ s = 12285.$$

4. Given n, r, and s, to find a and l.

$$Ans.\ a = \frac{s(r - 1)}{r^n - 1},\ l = \frac{(r - 1)sr^{n-1}}{r^n - 1}.$$

5. Given $n = 6$, $r = 3$, and $s = 1820$, to find a and l.

$$Ans.\ a = 5,\ l = 1215.$$

6. Given a, l, and n, to find r and s.

$$Ans.\ r = \sqrt[n-1]{\frac{l}{a}},\ s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$$

7. Given $a = 5$, $l = 1215$, and $n = 6$, to find r and s.

$$Ans.\ r = 3,\ s = 1820.$$

8. Given a, n, and s, to find r and l.

$$Ans.\ ar^n - sr = a - s,\ l(s - l)^{n-1} = a(s - a)^{n-1}.$$

From these equations r and l may be found, when the numerical values of a, n, and s are given.

9. Given l, n, and s, to find a and r.
Ans. $a(s-a)^{n-1} = l(s-l)^{n-1}$, $(s-l)r^n - sr^{n-1} = -l$.

10. Given a, r, and l, to find s and n.

$$\textit{Ans.}\ s = \frac{lr-a}{r-1},\ r^{n-1} = \frac{l}{a}.$$

To find n raise r to a power which is equal to $\frac{l}{a}$. The exponent of this power will be $n-1$.

11. Given $a = 1$, $r = 2$, and $l = 512$, to find s and n.
Ans. $s = 1023$, $n = 10$.

12. Given a, l, and s, to find r and n.

$$\textit{Ans.}\ r = \frac{s-a}{s-l},\ \left(\frac{s-a}{s-l}\right)^{n-1} = \frac{l}{a}.$$

13. Given $a = 4$, $l = 12500$, and $s = 15624$, to find r and n.
Ans. $r = 5$, $n = 6$.

14. Given a, r, and s, to find l and n.

$$\textit{Ans.}\ l = \frac{a + s(r-1)}{r},\ r^n = \frac{a + s(r-1)}{a}$$

15. Given $a = 1$, $r = 2$, and $s = 1023$, to find l and n.
Ans. $l = 512$, $n = 10$.

16. Given l, r, and s, to find a and n.

$$\textit{Ans.}\ a = lr - s(r-1),\ r^{n-1} = \frac{l}{lr - s(r-1)}.$$

17. Given $l = 192$, $r = 2$, and $s = 360$, to find a and n.
Ans. $a = 24$, $n = 4$.

GEOMETRIC MEAN.

332. The **Geometric Mean** of n quantities is the n^{th} root of their product. Thus, the geometric mean of 1, 3, 12, and 36 is $\sqrt[4]{1 \times 3 \times 12 \times 36} = 6$

333. To find the geometric mean of the terms of a G. P.

Denote the geometrical mean by m; then, by definition,

$$m = \sqrt[n]{\sqrt{(al)^n}} = \sqrt{al} \quad \textbf{(330)}.$$

Thus, the geometric mean of the terms of a G. P. whose extremes are 3 and 243 is $\sqrt{3 \times 243} = 27$.

334. To find a and l, when m, r, and n are given.

$$a = \frac{l}{r^{n-1}} \ \textbf{(326)}; \text{ whence, } a^2 = \frac{al}{r^{n-1}} = \frac{m^2}{r^{n-1}} \ \textbf{(330)};$$

whence, $$a = \frac{m}{\sqrt{r^{n-1}}}.$$

Again, $l = ar^{n-1}$; whence, $l^2 = alr^{n-1} = m^2r^{n-1}$;

whence, $$l = m\sqrt{r^{n-1}}.$$

EXERCISES.

335. 1. Write a G. P. of three terms, whose geometric mean is xy, and ratio $\frac{y}{x}$.

$$a = \frac{m}{\sqrt{r^{n-1}}} = \frac{xy}{\sqrt{\left(\frac{y}{x}\right)^2}} = x^2;$$

hence, the second term $= x^2 \times \frac{y}{x} = xy$;

and the third term $= xy \times \frac{y}{x} = y^2$.

Hence, x^2, xy, y^2 is the required progression.

2. Write a G. P. of four terms, whose geometric mean is $\sqrt{xy}$, and ratio $\frac{y}{x}$.

$$a = \frac{m}{\sqrt{r^{n-1}}} = \frac{\sqrt{xy}}{\sqrt{\left(\frac{y}{x}\right)^3}} = \frac{x^2}{y};$$

hence, the second term $= \frac{x^2}{y} \times \frac{y}{x} = x;$

the third term $= x \times \frac{y}{x} = y;$

and the fourth term $= y \times \frac{y}{x} = \frac{y^2}{x}.$

Hence, $\frac{x^2}{y}, x, y, \frac{y^2}{x}$ is the required progression.

3. The sum of three numbers in G. P. is 13, and the sum of their squares is 91. What are the numbers?

Ans. 1, 3, 9.

Denote the numbers by x^2, xy, and y^2.

4. The sum of the first and third of four numbers in G. P. is 10, and the sum of the second and fourth is 30. What are the numbers? *Ans.* 1, 3, 9, 27.

Denote the numbers by $\frac{x^2}{y}$, x, y, and $\frac{y^2}{x}$.

5. The sum of three numbers in G. P. is 210, and the difference between the extremes is 90. What are the numbers? *Ans.* 30, 60, 120.

6. The sum of the terms of a G. P. of four terms is 15, and if the last term be divided by the sum of the means, the quotient will be $1\frac{1}{3}$. What is the progression?

Ans. 1, 2, 4, 8.

7. The product of the terms of a G. P. of three terms is 64, and the sum of the squares of the extremes is 68. What is the progression? *Ans.* 2, 4, 8.

8. The product of three numbers in G. P. is 27, and the sum of their cubes is 757. What are the numbers?

Ans. 1, 3, 9.

CHAPTER XIV.

CONTINUED FRACTIONS.

336. A **Continued Fraction** is a complex fraction of the form of

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}}$$

in which a, b, c, d, etc., are positive integers.

337. A **Converging Fraction**, or a **Convergent**, is the result obtained by considering a certain number of the simple fractions, beginning at the first, and neglecting all that follow. Thus, the first convergent of the continued fraction given in Art. **336** is $\frac{1}{a}$; the second convergent is the simple fraction which is equivalent to $\cfrac{1}{a+\cfrac{1}{b}}$; and so on.

338. To simplify a terminating continued fraction.

EXERCISES.

1. Simplify $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}$.

$$3+\frac{1}{4}=\frac{13}{4}; \qquad \therefore\ \cfrac{1}{3+\cfrac{1}{4}}=\frac{4}{13}.$$

$$2+\frac{4}{13}=\frac{30}{13}; \qquad \therefore \frac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}=\frac{13}{30}.$$

2. Simplify $\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2}}}.$ *Ans.* $\frac{7}{30}.$

3. Simplify $1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9}}}}.$ *Ans.* $\frac{1380}{1051}.$

3. Simplify $\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}.$ *Ans.* $\frac{bc+1}{a(bc+1)+c}.$

339. To find approximate values of a continued fraction which does not terminate.

EXERCISES.

1. Find approximate values of $\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\text{etc.}}}}}$

$$\frac{1}{2} = \text{the first convergent;}$$

$$\cfrac{1}{2+\cfrac{1}{4}}=\frac{4}{9} = \text{the second convergent;}$$

$$\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2}}}=\frac{9}{20} = \text{the third convergent; and so on.}$$

The convergents of an odd order are greater, and those of an even order are less, than the value of the continued fraction. Thus, $\frac{1}{2}$ is too great, because the denominator 2 is too small; $\frac{4}{9}$ is too small, because $2 + \frac{1}{4}$ is too great; and so on.

2. Find the fourth convergent of $\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \text{etc.}}}}}$

Ans. $\frac{8}{11}$.

3. Find approximate values of $3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \text{etc.}}}}$

Ans. 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, ...

4. Find approximate values of

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \text{etc.}}}}}}$$

Ans. 1, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$, $\frac{99}{70}$, ...

340. To reduce a quantity to an expression containing a continued fraction.

1. Reduce $\frac{17}{15}$ to an expression containing a continued fraction.

$$\frac{17}{15} = 1 + \frac{2}{15} = 1 + \cfrac{1}{\cfrac{15}{2}} = 1 + \cfrac{1}{7 + \cfrac{1}{2}}.$$

The integral part, 1, and the denominators, 7 and 2, are the quotients which would be obtained by the process of finding the g. c. d. of 17 and 15.

2. Reduce $\frac{15}{17}$ to a continued fraction. *Ans.* $\dfrac{1}{1+\dfrac{1}{7+\dfrac{1}{2}}}$.

3. Reduce $\sqrt{2}$ to an expression containing a continued fraction, and find approximate values.

$$\sqrt{2} = 1.4142136 + = \frac{14142136+}{10000000}.$$

Applying the process for finding the g. c. d., we find the quotients to be 1, 2, 2, 2, ... and 2 perpetually.

$$\therefore \quad \sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}$$

The approximate values are 1, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$, $\frac{99}{70}$, ... (**339**, 4).

4. Find approximate values for the ratio of 5 hours 48 min. 51 sec. to 24 hours. *Ans.* $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{39}{161}$, ...

5. Find approximate values of 3.1415926535...., which is the ratio of the circumference of a circle to its diameter. *Ans.* 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, ...

6. The length of the tropical year is 365.24224 days, and the moon's synodic period is 29.5305887 days. Find approximate values for the ratio of these numbers. *Ans.* 12, $\frac{25}{2}$, $\frac{37}{3}$, $\frac{99}{8}$, $\frac{136}{11}$, $\frac{235}{19}$, $\frac{4131}{334}$, ...

7. Find approximate values for the ratio of a meter to a yard. *Ans.* 1, $\frac{11}{10}$, $\frac{12}{11}$, ...

8. Find approximate values for the ratio of a kilometer to a mile. *Ans.* 1, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{8}$, ...

341. To solve an exponential equation by means of continued fractions.

An **Exponential Equation** is an equation in which the unknown quantity occurs in an exponent. Thus,

$$10^{2x} = 100 \text{ is an exponential equation.}$$

EXERCISES.

1. Solve $10^x = 5$.

If $x = 0$, $10^x = 1$; and, if $x = 1$, $10^x = 10$; hence, the value of x lies between 0 and 1.

Put $x = \frac{1}{x'}$; then $10^{\frac{1}{x'}} = 5$;

whence, $10 = 5^{x'}$.

If $x' = 1$, $5^{x'} = 5$; and, if $x' = 2$, $5^{x'} = 25$; hence, the value of x' lies between 1 and 2.

Put $x' = 1 + \frac{1}{x''}$; then

$$5^{1+\frac{1}{x''}} = 10;$$

whence, by dividing by 5,

$$5^{\frac{1}{x''}} = 2;$$

whence, $5 = 2^{x''}$.

If $x'' = 2$, $2^{x''} = 4$; and, if $x'' = 3$, $2^{x''} = 8$; hence, the value of x'' lies between 2 and 3.

Put $x'' = 2 + \frac{1}{x'''}$; then

$$2^{2+\frac{1}{x'''}} = 5;$$

whence, by dividing by 2^2,

$$2^{\frac{1}{x'''}} = \frac{5}{4};$$

whence, $$2 = \left(\frac{5}{4}\right)^{x'''}$$

If $x''' = 3$, $\left(\frac{5}{4}\right)^{x'''} < 2$; and, if $x''' = 4$, $\left(\frac{5}{4}\right)^{x'''} > 2$; hence, the value of x''' lies between 3 and 4.

Put $x''' = 3 + \frac{1}{x''''}$; then

$$\left(\frac{5}{4}\right)^{3+\frac{1}{x''''}} = 2;$$

whence, by dividing by $\left(\frac{5}{4}\right)^3$,

$$\left(\frac{5}{4}\right)^{\frac{1}{x''''}} = \frac{128}{125};$$

whence, $$\frac{5}{4} = \left(\frac{128}{125}\right)^{x''''}.$$

If $x'''' = 9$, $$\left(\frac{128}{125}\right)^{x''''} < \frac{5}{4};$$

and, if $x'''' = 10$, $$\left(\frac{128}{125}\right)^{x''''} > \frac{5}{4};$$

hence, the value of x'''' lies between 9 and 10.

$$\therefore \quad x''' = 3 + \cfrac{1}{9 + \text{etc.},}$$

$$x'' = 2 + \cfrac{1}{3 + \cfrac{1}{9 + \text{etc.},}}$$

$$x' = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{9 + \text{etc.},}}}$$

and
$$x = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{9 + \text{etc.},}}}} = .69892\ldots,$$

which is correct to four places.

The value of x which satisfies the equation $10^x = 5$ is called the *logarithm* of 5.

Solve:

2. $3^x = 100$. *Ans.* $x = 4.19+$.
3. $3^{2x} = 15$. *Ans.* $x = 1.23+$.
4. $10^x = 3$. *Ans.* $x = 0.477+$.
5. $10^x = 31.622777$. *Ans.* $x = 1.5$.
6. $10^x = 21.544347$. *Ans.* $x = 1\frac{1}{3}$.
7. $25^x = 125$. *Ans.* $x = 1.5$.
8. $64^x = 1024$. *Ans.* $x = 1\frac{2}{3}$.

CHAPTER XV.

LOGARITHMS.

342. The value of x in the equation $10^x = n$ is called the **Logarithm** *of n to the base* 10, or, simply, *the logarithm* of n, 10 being the base of the only system of logarithms in common use. Thus, the logarithm of 100 is 2, for $10^2 = 100$, and the logarithm of 5 is .6989, for it has been shown that the value of x in the equation $10^x = 5$ is .6989.

343. The **Characteristic** of a logarithm is the integral part of it, and the **Mantissa** is the fractional part, which is usually written as a decimal. Thus, the characteristic of 3.17609, which is the logarithm of 1500, is 3, and the mantissa is .17609.

344. To find the characteristic.

1. Every number having n figures in its integral part lies between 10^{n-1} and 10^n. Thus, 6 lies between 10^0 and 10^1, 29.3 lies between 10^1 and 10^2, 839.4 lies between 10^2 and 10^3, and so on.

Hence, the logarithm of a number having n figures in its integral part lies between $n - 1$ and n. Its characteristic must therefore be $n - 1$; that is,

The characteristic of the logarithm of a number having an integral part is ONE *less than the number of figures in that part.* Thus, the characteristics of the logarithms of 32, 713.54, 8.7168, 56452, 873607.9 are respectively 1, 2, 0, 4, 5.

2. Every number less than 1 and expressed as a decimal having n ciphers immediately after the decimal point lies

between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, or between 10^{-n} and $10^{-(n+1)}$ **(141)**. Thus, .3 lies between 1 and .1, or between 10^0 and 10^{-1}, .027 lies between .1 and .01, or between 10^{-1} and 10^{-2}, .0003504 lies between .001 and .0001, or between 10^{-3} and 10^{-4}, and so on.

Hence, the logarithm of a decimal fraction having n ciphers immediately after the decimal point lies between $-n$ and $-(n+1)$. It is therefore negative, and the integral part of it is $-n$. But for convenience the mantissa is always kept *positive;* hence, the integral part of the logarithm must be $-(n+1)$; that is,

The characteristic of the logarithm of a decimal fraction is negative and numerically ONE *greater than the number of ciphers immediately after the decimal point.* Thus, the characteristics of the logarithms of .3, .0076, 02535, .000708 are respectively -1, -3, -2, -4.

SCH. As the mantissa is always kept *positive* the sign $-$ is written *above* a negative characteristic, to show that it alone is negative. Thus, $\log .015 = \bar{2}.17609$, which is equivalent to $-2 + .17609$.

EXERCISES.

Find the characteristic of the logarithm of

1.	13.	*Ans.* 1.
2.	211.	*Ans.* 2.
3.	3000.	*Ans.* 3.
4.	.15.	*Ans.* -1.
5.	7.012.	*Ans.* 0.
6.	.003.	*Ans.* -3.
7.	3135.2.	*Ans.* 3.
8.	.000052.	*Ans.* -5.

345. Th. *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of the numbers.*

Suppose $m = 10^x,$

$n = 10^y,$

$p = 10^z;$

then $mnp = 10^{x+y+z}$

But, by definition, $\log m = x$, $\log n = y$, $\log p = z$, and $\log (mnp) = x + y + z;$

$$\therefore \quad \log (mnp) = \log m + \log n + \log p.$$

346. Th. *The logarithm of the quotient of two numbers is equal to the result obtained by subtracting the logarithm of the divisor from that of the dividend.*

Suppose $m = 10^x,$

and $n = 10^y;$

then $\frac{m}{n} = 10^{x-y}.$

But, by definition, $\log m = x$, $\log n = y$, and $\log \left(\frac{m}{n}\right) = x - y;$

$$\therefore \quad \log \left(\frac{m}{n}\right) = \log m - \log n.$$

347. Th. *The logarithm of any power of a number is equal to the product obtained by multiplying the logarithm of the number by the exponent of the power.*

Suppose $m = 10^x;$

then $m^r = (10^x)^r = 10^{rx}.$

$$\therefore \quad \log (m^r) = rx = r \log m.$$

348. Th. *The logarithm of any root of a number is equal to the quotient obtained by dividing the logarithm of the number by the index of the root.*

Suppose $$m = 10^x;$$

then $$\sqrt[r]{m} = \sqrt[r]{10^x} = 10^{\frac{x}{r}}.$$

$$\therefore \quad \log\left(\sqrt[r]{m}\right) = \frac{x}{r} = \frac{\log m}{r}.$$

EXERCISES.

349. 1. Given $\log 2 = .3010$, and $\log 3 = .4771$, to find the logarithms of 6, 18, 15, .125, and 6.75.

$$\text{Log } 6 = \log(3 \times 2) = \log 3 + \log 2 = .4771 + .3010 = .7781.$$

$$\text{Log } 18 = \log(6 \times 3) = \log 6 + \log 3 = .7781 + .4771 = 1.2552.$$

$$\text{Log } 15 = \log\left(3 \times \frac{10}{2}\right) = \log 3 + \log 10 - \log 2$$
$$= .4771 + 1 - .3010 = 1.1761.$$

$$\text{Log } .125 = \log\left(\frac{1}{2^3}\right) = \log 1 - 3 \log 2 = 0 - 3 \times .3010$$
$$= -.9030 = -1 + (1 - .9030) = -1 + .0970 = \bar{1}.0970.$$

$$\text{Log } 6.75 = \log\left(\frac{27}{4}\right) = \log\left(\frac{3^3}{2^2}\right) = 3 \log 3 - 2 \log 2$$
$$= 3 \times .4771 - 2 \times .3010 = .8293.$$

2. Given log 2, to find log 5. *Ans.* .6990.

3. Given log 2, to find log $\frac{1}{2}$. *Ans.* $\bar{1}.6990$.

4. Given log 3, to find log $\sqrt[3]{3}$, and log $33\frac{1}{3}$.
Ans. .1590, 1.5229.

5. Show that $\log \sqrt[3]{abc} = \frac{1}{3}(\log a + \log b + \log c)$.

6. Given log 20763 = 4.3173, to find the logarithms of 2.0763, 2076.3, .020763, and .0020763.
Ans. 0.3173, 3.3173, $\bar{2}.3173$, $\bar{3}.3173$.

7. Given log 20.912 = 1.3204, to find the numbers whose logarithms are $\bar{2}.3204$, 6.3204, $\bar{1}.3204$, and 4.3204.
Ans. .020912, 2091200, .20912, 20912.

350. A TABLE OF LOGARITHMS COMPLETE FOR FOUR PLACES.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	**41**
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	**38**
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	**35**
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	**32**
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	**30**
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	**28**
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	**26**
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	**25**
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	**24**
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	**22**
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	**21**
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	**20**
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	**19**
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	**19**
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	**18**
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	**17**
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	**16**
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	**16**
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	**15**
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	**15**
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	**14**
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	**14**
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	**13**
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	**13**
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	**13**
35	5441	5453	5465	5478	5490	5502	5515	5527	5539	5551	**12**
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	**12**
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	**12**
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	**11**
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	**11**
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	**11**
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	**11**
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	**10**
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	**10**
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	**10**
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	**10**
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	**9**
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	**9**
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	**9**
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	**9**
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	**9**
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	**9**
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	**8**
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	**8**
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	**8**
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	

TABLE OF LOGARITHMS—*Continued.*

No.	0	1	2	3	4	5	6	7	8	9	Diff.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	8
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	7
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	6
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9899	9894	9899	9903	9908	5
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	5
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	

The preceding table contains the mantissas of the logarithms of numbers from 1 to 1000 inclusive. The characteristics are not given, but they are easily supplied by the principles of Art. **344**.

EXERCISES.

351. 1. Find the logarithm of 6.

The characteristic of log 6 is 0, and the mantissa is the same as the mantissa of log 60; for

$$\log 6 = \log\left(\frac{60}{10}\right) = \log 60 - \log 10 = \log 60 - 1,$$

and the subtraction of 1 from log 60 does not change the mantissa. $\therefore$ log 6 = 0.7782.

2. Find the logarithm of 7. *Ans.* 0.8451.

3. Find the logarithm of .07.

The characteristic of log .07 is -2, and the mantissa is the same as the mantissa of log 7; for

$$\log .07 = \log\left(\frac{7}{100}\right) = \log 7 - \log 100 = \log 7 - 2,$$

and the subtraction of 2 from log 7 does not change the mantissa. $\therefore$ log .07 = $\bar{2}.8451$.

4. Find the logarithm of .009. *Ans.* $\bar{3}.9542$.

5. Find the logarithm of .2137.

The characteristic of log .2137 is -1.

To find the mantissa, look for 21 in the No. column; then run along the line toward the right till you come into the column headed 3, which is the third figure in the number .2137;

$$\begin{array}{lr} \text{take out log .2130} & = \bar{1}.3284 \\ \text{Diff. for 7, the fourth figure,} & = \underline{\quad 14} \\ \text{log .2137} & = \bar{1}.3298. \end{array}$$

Continue along the line till you come into the column headed Diff.; take out 20. This is the difference between log .2130 and log .2140; hence, the difference between log .2130 and log .2137 is 20 × .7 = 14. Add this to $\bar{1}.3284$. The sum, $\bar{1}.3298$, is the logarithm of .2137.

6. Find the number whose logarithm is 2.8531.

The mantissa is found at the right of 71 and in the column headed 3. The number of figures in the integral part of a number is one greater than the characteristic of its logarithm; hence, the number whose logarithm is 2.8531 is 713.

7. Find the number whose logarithm is .3142.

This logarithm lies between .3139 and .3160 in the table. The smaller of these logarithms, .3139, is in the column headed 6 and at the right of 20 in the No. column; hence, take out log 2.06 = .3139; also, tabular diff. = 21. By subtracting .3139 from the given logarithm, .3142, we find the difference to be 3.

Denote the fourth figure of the required number by x; then, as in Ex. 5,

$$21 \times \frac{x}{10} = 3;$$

whence,
$$x = \frac{3 \times 10}{21} = 1+.$$

Hence, the number whose logarithm is .3142 is 2.061.

The work may be abridged as follows:

$$\begin{array}{rl} & .3142 \\ \log 2.06 = & .3139 \\ \hline & 21\ )\ 30\ (\ 1. \end{array}$$

8. Find the number whose logarithm is $\bar{1}.5700$.

$$\begin{array}{rl} & \bar{1}.5700 \\ \log .371 = & \bar{1}.5694 \\ \hline & 12\ )\ 60\ (\ 5. \end{array}$$

Hence, the required number is .3715.

9. Find, by means of logarithms, the product of .3715 and .8913.

$$\begin{aligned} \text{Log } .3715 &= \bar{1}.5700 \\ \text{log } .8913 &= \underline{\bar{1}.9501} \\ \text{Sum} &= \bar{1}.5201 = \text{log of product } (\mathbf{345}). \end{aligned}$$

The number whose logarithm is $\bar{1}.5201$ is .3312, which is the product required.

10. Divide .7132 by 8.162.

$$\begin{aligned} \text{Log } .7132 &= \bar{1}.8532 \\ \text{log } 8.162 &= \underline{0.9118} \\ \text{Difference} &= \bar{2}.9414 = \text{log of quotient } (\mathbf{346}). \end{aligned}$$

The number whose logarithm is $\bar{2}.9414$ is .08738, which is the quotient required.

11. Find the fourth power of .328 (**347**).

12. Find the cube root of .000512 (**348**).

352. To solve an exponential equation by means of logarithms.

EXERCISES.

1. Solve $10^x = 75$.

It is evident, from the definition of a logarithm, that the value of x in this equation is the logarithm of 75;

$$\therefore \quad x = 1.8751.$$

2. Solve $18^x = 125$.

Taking the logarithm of each member of this equation,

$$x \log 18 = \log 125 \ \ (\mathbf{347});$$

whence, $$x = \frac{\log 125}{\log 18} = \frac{2.0969}{1.2553} = 1.670+.$$

Solve:

3. $3^x = 81.$ *Ans.* $x = 4.$

4. $5^{2x} = 10000.$ *Ans.* $x = 2.861+.$

5. $4^x = 8.$ *Ans.* $x = 1.5.$

6. $10^x = 3.$ *Ans.* $x = 0.4771+.$

7. $\left(\frac{7}{12}\right)^x = \frac{3}{4}.$ *Ans.* $x = 0.53+.$

8. $5^x = \frac{2}{3}.$ *Ans.* $x = -0.25+.$

9. $2^x = \frac{1}{3}.$ *Ans* $x = -1.58+.$

10. $ab^x = c.$ *Ans.* $x = \frac{\log c - \log a}{\log b}.$

11. $ab^{\frac{1}{x}} = c.$ *Ans.* $x = \frac{\log b}{\log c - \log a}.$

12. $5^{2+x} = 10000.$ *Ans.* $x = 3.722+.$

13. $8^{\frac{1}{1+x}} = 4$ *Ans.* $x = .5.$

14. $2.718^x = 4.$ *Ans.* $x = 1.386+.$

15. $a^{-x} = b.$ *Ans.* $x = -\frac{\log b}{\log a}.$

16. $a^{-\frac{1}{x}} = b.$ *Ans.* $x = -\frac{\log a}{\log b}.$

17. $a^{\sqrt{x}} = b.$ *Ans.* $x = \left(\frac{\log b}{\log a}\right)^2.$

18. $a^{2x} - b = 2ca^x.$ *Ans.* $x = \frac{\log\left(c \pm \sqrt{b + c^2}\right)}{\log a}.$

GEOGRAPHY.

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This work certainly deserves its name in the best sense. Though complete, it is not, like most others which bear the same title, *cumbersome.* These authors excel in clear, lucid demonstrations, teaching the science pure and simple, yet not ignoring convenient methods and practical applications.

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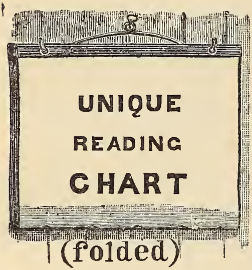

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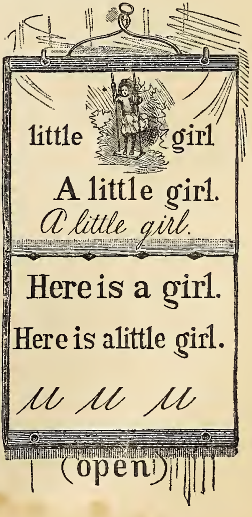
little girl
A little girl.
A little girl.
Here is a girl.
Here is alittle girl.
u u u
(open)

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